Analysis of Single-Shuffle Trick Failure Rates

with binomial estimations of human splitting and insertion

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Splitting the deck

Stochastically splitting the deck into two piles involves only the sampling of a single random number between 2 and 50; the number of cards to be placed in one of the piles (we restrict the minimum pile size to be 2 here, since piles of size 1 introduces a greater chance - and thus a more complex calculation - of trick failure).

Let A and B be the number of cards in our left and right piles respectively (such that $A + B = 52$), where A takes the value of our random number, which we choose to sample from a Binomial distribution with an n parameter of 48 (traditionally the number of trials, which we remap to the number of discrete split choices possible) and a p parameter of $\frac{1}{2}$ (to center the peak of most likely split around the perfect center of the deck).

The probability of having A cards in our left pile becomes

$$\begin{aligned} P_{\text{split}}(A) &= \binom{48}{A} \times \left(\frac{1}{2}\right)^{A} (1 - \frac{1}{2})^{48-A} \\ &= \frac{48!}{A!(48-A)!} \times \frac{1}{2^{48}} \end{aligned} \tag{1}$$

Riffling the piles

We imagine the riffle of the piles back into a complete deck as the gradual construction of a new pile, by the sequential selection and removal of a single card from the top of either the left and right piles, until both are depleted. In lieu of a (more) naive uniform selection between the piles, we let the probability of the next card being selected from either pile equal the ratio of cards remaining in that pile to the sum of those in both (as if we were choosing uniformly between all remaining cards).

Let the probability of selecting a card (a_n) from the left pile when the left and right piles contain A and B cards respectively be denoted

$$P_{A,B}(a_n) = \frac{A}{A+B},$$

whilst under the same conditions, the probability of selecting a card (b_n) from the right pile be

$$P_{A,B}(b_n) = \frac{B}{A+B}.$$

Note that due to symmetry, we can imagine (for later convenience) that our riffle is performed such that the first cards selected from the two piles remain on the top of the final complete deck. i.e. we append a newly selected card (from one of the two piles) to the *bottom* of the forming deck.

For example, the probability of having a final complete deck where the first card (a_1) originates from the left pile and the next n cards ($b_1, ..., b_n$) originate from the right, is equal to initially selecting a card from the left pile then selecting n from the right.

$$
\begin{aligned}
P_{\text{riffle}}(a_1, b_1, ..., b_n) &= P_{A,B}(a_1) \times P_{A-1,B}(b_1) \times P_{A-1,B-1}(b_2) \times ... \times P_{A-1,B-(n-1)}(b_n) \\
&= \frac{A}{52} \times \frac{B}{51} \times \frac{B-1}{50} \times ... \times \frac{B-(n-1)}{52-n} \\
&= A \times \frac{B!}{(B-n)!} \times \frac{(52-(n+1))!}{52!} \\
&= A \times \frac{B!}{(B-n)!} \times \frac{(51-n)!}{52!}. \qquad (2)
\end{aligned}
$$

Similarly, achieving a deck with a top card originating from the right pile and the next n cards origingating from the left pile has probability

$$
P_{\text{riffle}}(b_1, a_1, ..., a_n) = B \times \frac{A!}{(A-n)!} \times \frac{(51-n)!}{52!}.
$$

Misplacing the top card

We model the misplacement of the top card to the approximate middle of the deck as the sampling of a random number between 1 and 50 (inclusive); the indices of the spaces between the bottom 51 cards. We again choose to sample from a Binomial distribution with n parameter 49 and p parameter $\frac{1}{2}$. The probabilty of inserting the card in the i^{th} space becomes

$$
\begin{aligned}
P_{\text{insert}}(i) &= \binom{49}{i} \times \left(\frac{1}{2}\right)^i (1 - \frac{1}{2})^{49-i} \\
&= \frac{49!}{i!(49-i)!} \times \frac{1}{2^{49}} \qquad (3)
\end{aligned}
$$

Failing the trick

After the initially ordered deck (to be imagined as ascending numbers 1 to 52) is randomly split into two piles (each with a size no smaller than two), imperfectly riffle shuffled back into a complete deck and has its top card misplaced to the approximate middle, the trick is performed by repeatedly revealing the top card and placing it atop the pile which features its value minus 1, otherwise placed in a new pile.

For a general number of riffle shuffles, the trick is successful when only a single one-card pile exists, and that card is the previous top. The trick is otherwise considered unsuccessful if there are multiple one-card piles (caused by unfortunate riffling), or no one-card piles exist (caused by unfortunate top misplacement); in either case, the previous top can not be uniquely identified.

However, a single riffle can not result in the circumstances required to cause multiple one-card piles, which is a result of the partitions of orderedness being divided into their minimum size (a single card).

Consider a deck of order $a_1, ..., a_n, b_1, ...b_m$ (where unimportantly, $m + n = 52$), which is split after the n^{th} card into piles $a_1, ..., a_n$ and $b_1, ..., b_m$ (in our left and right hands respectively). We observe that when riffling these piles back into a complete deck, it is impossible to altar the relative order among the cards of each pile. For example, the final deck could read

$$a_1, b_1, b_2, b_3, a_2, a_3...$$

but could not be ordered as

$$a_1, a_3, b_1, a_2, b_2...$$

since the relative order of a_2 and a_3 has been altared, which is impossible under the riffle operation. If the deck was distributed / stacked into the individual piles without a prior top misplacement, then the lack of disruption to the relative order in each partition means there will never be a one-card pile (unless we relaxed the originally split pile size minimum to 1, hence why we don't).

Thus for a single riffle, the trick is only unsuccessful when the misplacement does not result in a one-card pile and therefore did not break the relative order of its partition; this requires that on its journey from the top to its final location, it did not pass any other cards from its partition.

$$a_1, b_1, ..., b_n... \to b_1, ..., b_n, a_1...$$

or

$$b_1, a_1, ..., a_n... \to a_1, ..., a_n, b_1...$$

These failure cases are not excluded by any initial A, B split (for $A, B > 1$), are enabled by a particular set of riffle outcomes (as above) and are observed when the top card is misplaced within the riffle-dictated threshold.

Axiomatically quantifying failure

With our top card (a_1) already removed, having a remaining deck of the form $b_1, ..., b_n, a_2...$ provides n opportunities to insert the card without violating a_i order and thus failing the trick. The probability of failure under these constraints is therefore

$$P_{\text{failed insert}} = \sum_{i=1}^{n} P_{\text{insert}}(i) = \sum_{i=1}^{n} \frac{49!}{i!(49-i)!} \times \frac{1}{2^{49}}. \qquad \text{(by eq 3)}$$

and is equal to that of the same situation where a and b are interchanged (inserting a top b_1 card into $a_1, ..., a_n, b_2...$ cards).

For a given split A and B of the initially ordered deck, after a riffle shuffle, following a top card of a_1 there may be anywhere from 0 to B contiguous cards from the right pile, afterwhich (or inbetween) an immediate insertion results in failure. There might also be a top b_1 then contiguous 0 to A cards from the left pile. Thus, the probability of failure under these contrains becomes

$$\begin{aligned}
P_{\text{failed riffle insert}} &= \sum_{n=1}^{B} \left(P_{\text{riffle}}(a_1, b_1, ..., b_n) \times P_{\text{failed insert}}\right) + \sum_{n=1}^{A} \left(P_{\text{riffle}}(b_1, a_1, ..., a_n) \times P_{\text{failed insert}}\right) \\
&= \sum_{n=1}^{B} \left(A \times \frac{B!}{(B-n)!} \times \frac{(51-n)!}{52!} \times \sum_{i=1}^{n} \frac{49!}{i!(49-i)!} \times \frac{1}{2^{49}} \right) + \\
&\quad \sum_{n=1}^{A} \left(B \times \frac{A!}{(A-n)!} \times \frac{(51-n)!}{52!} \times \sum_{i=1}^{n} \frac{49!}{i!(49-i)!} \times \frac{1}{2^{49}} \right) \qquad \text{(by eq 2)} \\
&= \sum_{n=1}^{52-A} \left(A \times \frac{(52-A)!}{(52-A-n)!} \times \frac{49!(51-n)!}{52!} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \times \frac{1}{2^{49}} \right) + \\
&\quad \sum_{n=1}^{A} \left((52-A) \times \frac{A!}{(A-n)!} \times \frac{49!(51-n)!}{52!} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \times \frac{1}{2^{49}} \right) \\
&\qquad \text{(since } A + B = 52\text{)} \\
&= \sum_{n=1}^{52-A} \left(\frac{A}{2^{49}} \times \frac{(52-A)!}{(52-A-n)!} \times \frac{(51-n)!}{52 \times 51 \times 50} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \right) + \\
&\quad \sum_{n=1}^{A} \left(\frac{52-A}{2^{49}} \times \frac{A!}{(A-n)!} \times \frac{(51-n)!}{52 \times 51 \times 50} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \right).
\end{aligned}$$

To find the total chance of failure from the very beginning of the trick, we simply factor in the probability of achieving that given A, B split, summed across all possible splits.

$$\begin{aligned}
P_{\text{failed trick}} &= \sum_{A=2}^{50} \left(P_{\text{split}}(A) \times P_{\text{failed riffle insert}}\right) \\
&= \sum_{A=2}^{50} \frac{48!}{A!(48-A)!} \times \frac{1}{2^{48}} \times \Bigg[\\
&\qquad \sum_{n=1}^{52-A} \left(\frac{A}{2^{49}} \times \frac{(52-A)!}{(52-A-n)!} \times \frac{(51-n)!}{52 \times 51 \times 50} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \right) + \\
&\qquad \sum_{n=1}^{A} \left(\frac{52-A}{2^{49}} \times \frac{A!}{(A-n)!} \times \frac{(51-n)!}{52 \times 51 \times 50} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \right) \Bigg] \\
&\qquad \text{(by eq 1)} \\
&= \frac{48!}{16575 \times 2^{100}} \sum_{A=2}^{50} \frac{1}{A!(48-A)!} \times \Bigg[\\
&\qquad \sum_{n=1}^{52-A} \left(A \times \frac{(52-A)!(51-n)!}{(52-A-n)!} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \right) + \\
&\qquad \sum_{n=1}^{A} \left((52-A) \times \frac{A!(51-n)!}{(A-n)!} \times \sum_{i=1}^{n} \frac{1}{i!(49-i)!} \right) \Bigg] \\
&= \frac{5291535747803275749786 48483}{2188677989456552326021651628 03200} \\
&\approx 2.41769 \times 10^{-6}
\end{aligned}$$

Empirically quantifying failure

We can also calculate an empirical quantification of the probability of failure by simulating the trick with stochastic decisions following the same distributions earlier defined, and taking the ratio of the failed simulations to the total performed. In the following code, the global constants `SHUFFLES` and `SIM_TRIALS` may be altared to modulate which shuffle-numbers (how many times the deck is riffle-shuffled each trick, ≥ 0) are simulated and how many times each are simulated in assessing their corresponding rate of failure. The global constant `BINOMIAL_PROB` can be edited to shift the most likely neighborhood of human card splitting / insertion, though this will cause divergence of the simulation from the axiomatically quantified scenario above.

```
import math
import random
import numpy

'total cards in the complete deck (CAN NOT BE CHANGED)'
DECK_SIZE = 52

'sets - as a ratio of the random var domain - the peak of highest probability location'
BINOMIAL_PROB = 0.5

'list of number-of-shuffles to simulate (serially)'
SHUFFLES = [1]

'number of trials to simulate the trick for each shuffle number'
SIM_TRIALS = 100000000

'number of simulations after which to print progress of computation (for each shuffle number)
PRINT_THRESHOLD = 0.1 * SIM_TRIALS

'samples a random number in [2, 50] from Binomail'
def sample_split():
    return 2 + numpy.random.binomial(48, 0.5)

'samples a random number in [1, 50] from Binomial'
def sample_insert():
    return 1 + numpy.random.binomial(49, 0.5)

# perform simulation(s) for each shuffle number
for shuffles in SHUFFLES:

    total_fails = 0

    print_counter = 0

    # simulate the trick SIM_TRIALS times...
    for sim in range(SIM_TRIALS):

        # print at designated progress points (for each number of shuffles)
        print_counter += 1
        if (print_counter >= PRINT_THRESHOLD):
            print_counter = 0
            print str((100*(sim+1))/SIM_TRIALS) + "%"
```

```
    cards = range(1, DECK_SIZE + 1)

    # perform the shuffle 'shuffles' times
    for _ in range(shuffles):

        new_cards = [0]*DECK_SIZE
        split_index = sample_split() + 1
        left, right = 0, split_index

        # merge splits by sequential pile selection, uniform over cards
        for z in range(0, DECK_SIZE):
            prob_left = (float(split_index - left)/
                         (DECK_SIZE - right + split_index - left))
            if (random.random() <= prob_left):
                new_cards[z] = cards[left]
                left += 1
            else:
                new_cards[z] = cards[right]
                right += 1
        cards = new_cards

    # misplace the top card to a randomly (Gaussian) selected index
    top_card = cards[0]
    insert_index = sample_insert()
    for y in range(insert_index):
        cards[y] = cards[y + 1]
    cards[insert_index] = top_card

    # form piles of contiguous cards dealt from the deck top
    piles = [[]]
    for card in cards:
        piled = False
        for pile in piles:
            if (len(pile) == 0) or (card == pile[-1] + 1):
                pile.append(card)
                piled = True
                break
        if not piled:
            piles.append([card])

    # detect failure by multiple single-card piles, or no presence of top-card pile
    failure = False
    found = False
    for pile in piles:
        if (len(pile)==1):
            if (pile[0] == top_card):
                found = True
            else:
                failure = True
                break
    if not found:
        failure = True

    # record failure
    if failure:
```

```
            total_fails += 1

    # print the empirical probabiliy of failure for this number of shuffles
    print (str(shuffles) + " shuffle" + ('s:' if shuffles!=1 else ': ') +
           " p = " + str(float(total_fails)/SIM_TRIALS))
```

Simulating between 0 and 7 riffle shuffles performed per trick with each trick being simulated 100K times, the probability of failure presented itself as:

```
0 shuffles: p = 0.0
1 shuffle:  p = 0.0
2 shuffles: p = 0.00175
3 shuffles: p = 0.10594
4 shuffles: p = 0.77213
5 shuffles: p = 0.98985
6 shuffles: p = 0.99944
7 shuffles: p = 0.99993
```

For 100M simulations of a single riffle shuffle, the empirical probability of failure was approximately:

```
1 shuffle:  p = 5.6e-7
```

References

[1] S. S. Roy, F. Vercauteren, I. Verbauwhede, *High Precision Discrete Gaussian Sampling on FPGAs*, ESAT/SCD-COSIC and iMinds, KU Leuven , **p3**
`http://homes.esat.kuleuven.be/~fvercaut/papers/SAC13.pdf`