Meeting 31/7/15

- Derive linear pendulum transfer functions (only gravitational coupling one found) ☑
- Read MAGPI paper ☑

31/7/15

Disclaimer: Monash's linear time series mathematics unit is confined to discrete-time analysis, so I'm probably mis-using its notation for continuous-time.

The *transfer function* for a linear time-invariant system (a continuous filter) T gives the relation between frequencies of oscillations of the system output and that of the input oscillations.

$$H(s) = \frac{\mathcal{L}\{(T \circ x)(t)\}}{\mathcal{L}\{x(t)\}}$$

The transfer function can be deduced directly from the ODE describing the system[1]. For example, a damped (with dampening parameter ν), driven (sinusoidally, with forcing amplitude f at frequency ω_D) pendulum has dimensionless equation of motion[2]

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \nu\frac{\mathrm{d}\theta}{\mathrm{d}t} + \sin(\theta) = f\cos(\omega_D t).$$

Here, the input is $x(t) = f\cos(\omega_D t)$ and the output is $(T \circ x)(t) = \theta(t)$.

For my sanity (I'm deathly afraid of the Laplace transform of $\sin(\theta(t))$), we apply the small angle approximation[3] $\sin(\theta) \approx \theta$, yielding

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \nu\frac{\mathrm{d}\theta}{\mathrm{d}t} + \theta \approx f\cos(\omega_D t).$$

We now take the Laplace transform of both sides (assuming initial conditions $\theta(0) = \theta_0$, $\theta'(0) = 0$; at rest at some angle), allowing us to collect $\Theta(s) = \mathcal{L}\{\theta(t)\}$ terms and express the transfer function as $\frac{\mathcal{L}\{\theta(t)\}}{\mathcal{L}\{x(t)\}}$.

$$\begin{aligned} & s^2\Theta(s) - s\theta(0) - \theta'(0) + \nu\left(s\Theta(s) - \theta(0)\right) + \Theta(s) && = \mathcal{L}\{x(t)\} \\ = & (s^2 + \nu s + 1)\Theta(s) - (s + \nu)\theta_0 && = f\frac{s}{s^2 + \omega_D^2}. \end{aligned}$$

Rearranging, we see

$$H(s) = \frac{\Theta(s)}{\mathcal{L}\{x(t)\}} = \frac{(s+\nu)\theta_0}{s^2 + \nu s + 1}\frac{s^2 + \omega_D^2}{fs}.$$

[1] **transferODE**.
[2] **penODE**.
[3] **smallAngle**.

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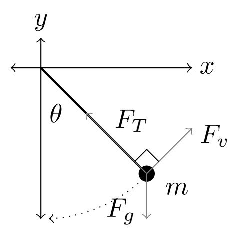
Listing 1: Deriving the equation of motion in terms of $\theta(t)$ via force resolution (Newton's 2nd law)

```

r[t_] := {l Sin[\[Theta][t]], -l Cos[\[Theta][t]]}

v[t_] := r'[t]

a[t_] := v'[t]

Subscript[F, g][t_] := {0, -m g}

Subscript[F, v][t_] := - \[Alpha] v[t]

Subscript[F, T][t_] := {-T Sin[\[Theta][t]], T Cos[\[Theta][t]]}

Subscript[F, D][t_] := {Subscript[F, Dx][t], Subscript[F, Dy][t]}

eqn = Eliminate[
Subscript[F, g][t] + Subscript[F, v][t] + Subscript[F, D][t] +
Subscript[F, T][t] == m a[t], T]

Simplify[eqn]
```

The chosen goemetric quantities are best visualized.



The final Mathematica statement yields 2nd order non-linear ODE (confirmed from here: http://www.math.cornell.edu/~hubbard/pendulum.pdf)

$$lm\theta''(t) + \alpha l\theta'(t) + gm\sin(\theta(t)) = F_{\mathrm{D}x}(t)\cos(\theta(t)) + F_{\mathrm{D}y}(t)\sin(\theta(t))$$

where the driving force is kept in generality: $\vec{F_D} = (F_{\mathrm{D}x}, F_{\mathrm{D}y})$. Notice the right hand side is merely the projection of the driving force along the arc of motion. For now, we'll represent this as $F_{\mathrm{D}\|\dot{\theta}}$, which implies a driving force parallel to $\vec{x}$ under the small angle approximation. We also now apply the small angle approximation ($\forall\ |\theta| \ll 1, \sin(\theta) \approx \theta$) to obtain the ODE

$$lm\theta''(t) + \alpha l\theta'(t) + gm\theta(t) = F_{\mathrm{D}\|\dot{\theta}}(t).$$

(I have hidden the dependence of the driving force on θ because we can could choose a driving force parallel to $\dot{\theta}$, cancelling the dependence of the ODE's RHS on θ and restoring it to be linear. But I don't want to do that prematurely).

Applying the Laplace transform to both sides yields

$$(gm + lms^2 + \alpha ls)\mathcal{L}\{\theta(t)\} - lm\theta'(0) - lms\theta(0) - \alpha l\theta(0) = \mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}.$$

If we assume typical initial conditions of $\theta(0) = \theta_0 \neq 0$, $\theta'(0) = 0$, then our ODE in Laplace space is

$$(gm + lms^2 + \alpha ls)\mathcal{L}\{\theta(t)\} - l\theta_0(ms + \alpha) = \mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}.$$

Is it correct here to rearrange for the transfer function definition $\frac{\mathcal{L}\{\theta(t)\}}{\mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}}$ which will remain in terms of $\mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}$? Assuming so...

$$F(s) = \frac{\mathcal{L}\{\theta(t)\}}{\mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}} = \frac{1 + l\theta_0(ms + \alpha)}{(gm + lms^2 + \alpha ls)\mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}}.$$

If we now assume a sinusoidal driving force which always applies parallel (or anti-parallel) to the pendulum's velocity $F_{\mathrm{D}\|\dot{\theta}(t)} = \beta\cos(\omega t)$, then our transfer function becomes

$$F(s) = \frac{1 + l\theta_0(ms + \alpha)(s^2 + \omega^2)}{(gm + lms^2 + \alpha ls)s\beta}$$

Derailment: having parameters of the frequency of a driving force adjustable to change the entire $F(s)$ function makes no sense, since different s values should already reflect different driving parameters. Understanding of transfer function is kill, shutting downnnnn.

4/8/15

Apparently initial conditiona are *transient* and may be ignored[4]. *I know understand that there is no direct conception of time in the transfer function; its value represents the response after an infinite time after which the initial conditions have become completely negligable.* For general initial conditions then, our ODE in Laplace space is

$$(gm + lms^2 + \alpha ls)\mathcal{L}\{\theta(t)\} = \mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}$$

corresponding to a transfer function

$$F(s) = \frac{\mathcal{L}\{\theta(t)\}}{\mathcal{L}\{F_{\mathrm{D}\|\dot{\theta}}(t)\}} = \frac{1}{gm + lms^2 + \alpha ls}.$$

[4] **transferConds**.

Meeting 7/8/15

- Derive linear pendulum transfer functions (seismic and g-coupling) via the Lagrangian, in a non-dimensional setting, in terms of the critical frequency and through fourier transforms ☑
- Plot transfer functions ☐
- Derive transfer functions (seismic and g-coupling) of torsion pendulum (through the Lagrangian, preferably) ☐

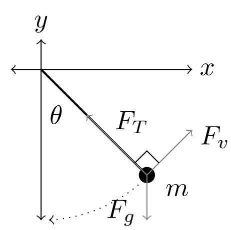
7/8/15

** how do I encode non-conservative forces in the Lagrangian formalism?
http://www.quora.com/Classical-Mechanics/What-is-the-Lagrangian-of-a-non-conservative-force
http://wikis.controltheorypro.com/index.php?title=Lagrange_Equations_of_Motion_for_NonConservative_Forces
https://www.kvi.nl/~scholten/AAM/Friction-Lagrange.pdf

Hesitantly using suspiciously simple[5]

Please forgive my use of character $\mathcal{L}$ for the Lagrangian, where previously used as the Laplace transform.

A linear pendulum is best described by the generalized coordinate $\theta(t)$.



At any time, the kinetic energy is given by

$$T(t) = \frac{1}{2}m|\vec{v}(t)|^2 = \frac{1}{2}ml^2\dot{\theta}^2(t).$$

Since neither our driving (assumably) or dampening forces are conservative, they have no associated potential functions. The potential is then simply $V(t) = -mg\cos(\theta(t))$ and the Lagrangian

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mg\cos(\theta).$$

For now, ignore the driving force and consider only the equation of motion due to the dampening $F_D = -\alpha l\dot{\theta}$. One source suggests I use Euler-Lagrange equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = F(\theta)$, but this yields $glm\sin(\theta(t)) + l^2m\theta''(t) + \alpha l\theta'(t) = 0$, which has observably too small an exponent on the l coefficient of $\theta'(t)$ (it should be 2, allowing us to divide everything by l, and have an $\alpha l\dot{\theta}$ remain).

[5] **nonconservLagrangian**.

Two other sources claim that for friction forces that are proportional to the velocity in particular can be substituted into equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\mathrm{d}F_D}{\mathrm{d}\dot{\theta}}$, but this incorrectly removes all dependence on θ' from our ODE.

https://en.wikipedia.org/wiki/Lagrangian_mechanics#Extensions_to_include_non-conservative_forces
Another source suggests to create (for viscous friction functions, such as our dampening) the Rayleigh dissipation function $D = \frac{1}{2}\alpha\dot{\theta}^2$, to substitute into equation $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = -\frac{\mathrm{d}D}{\mathrm{d}\dot{\theta}}$ but this still fails to include the required l^2 coefficient of $\dot{\theta}$.

Have mercy on me, non-conservative forces!

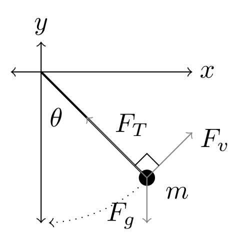
8/8/15

According to http://www.phys.uri.edu/gerhard/PHY520/mln9.pdf, the dissipative force has form

$$R = -\alpha v(\dot{q})\frac{\partial v(\dot{q})}{\partial \dot{q}}$$

meanwhile (http://wikis.controltheorypro.com/index.php?title=Lagrange_Equations_of_Motion_for_NonConservative_Forces) general non-conservative forces are left unchanged (in terms of the generalized coordinate q).

A linear pendulum is best described by the generalized coordinate $\theta(t)$.



At any time, the kinetic energy is given by

$$T(t) = \frac{1}{2}m|\vec{v}(t)|^2 = \frac{1}{2}ml^2\dot{\theta}^2(t).$$

Since neither our driving (assumably) or dampening forces are conservative, they have no associated potential functions. The potential is then simply the gravitational $V(t) = -mg\cos(\theta(t))$ and the Lagrangian

$$\mathcal{L}(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mg\cos(\theta).$$

Encapsulating both our viscous friction function $R = -\alpha v(\dot{\theta})\frac{\partial v(\dot{\theta})}{\partial \dot{\theta}}$ and driving force F, the Euler-lagrange equation has altered form

$$\begin{aligned} F(\theta) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - R \\ &= gml\sin(\theta) + l^2 m\ddot{\theta} + \alpha l^2\dot{\theta} \end{aligned}$$

But now if we try to divide both sides by l, our RHS is right but our LHS will be wrong!

$$\frac{F(\theta)}{l} = gm\sin(\theta) + lm\ddot{\theta} + \alpha l\dot{\theta}$$

I can't see any justification to lose the l on the LHS. Even though F is kept in complete generality, it's still the exact physical driving force. Eek!

After a discourse Csaba: the generalized force isn't necessarily a force at all, depending on coordinatization. For an angular coordinate, the generalized force is torque. Bless him!

9/8/15

http://ocw.mit.edu/courses/mechanical-engineering/2-003j-dynamics-and-control-i-spring-2007/lecture-notes/lec14.pdf

Non-conservative forces are introduced in the Lagrangian formalism for a system of n particles via the *generalized force*

$$Q(q)_j = \sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

where $\vec{F}_i$ is the total force on particle i and j identifies the coordinate.

Listing 2: Deriving the gravitational coupled pendulum's equation of motion via the Lagrangian

```

r[t_] := {l Sin[\[Theta][t]], -l Cos[\[Theta][t]]}
v[t_] := r'[t]
a[t_] := v'[t]

Subscript[F, v][t_] := - \[Alpha] v[t]
Subscript[F, D][t_] := {Subscript[F, Dx][t], Subscript[F, Dy][t]}

Q[t_] := Simplify[(Subscript[F, D][t] + Subscript[F, v][t]) . D[r[t], \[Theta][t]]]

L = 1/2 m l^2 \[Theta]'[t]^2 + m g l Cos[\[Theta][t]]

D[ D[ L, \[Theta]'[t] ], t] - D[L, \[Theta][t]] == Q[t]
```

After simplification, this yields ODE

$$lm\theta''(t) + \alpha l\theta'(t) + gm\sin(\theta(t)) = F_{\mathrm{D}x}(t)\cos(\theta(t)) + F_{\mathrm{D}y}(t)\sin(\theta(t)).$$

We again apply the small angle approximation and represent our RHS with a driving force which always points horizontal (still always orthogonal to the pendulum axis, due to the small angle).

$$lm\theta''(t) + \alpha l\theta'(t) + gm\theta(t) = F.$$

Taking the Fourier transform of both sides (and factorising) yields

$$(-lm\omega^2 - i\alpha l\omega + gm)\mathcal{F}\{\theta(t)\} = \mathcal{F}\{F\}.$$

10/8/15

Listing 3: Deriving the seismic response of the pendulum via the Lagrangian

```

s[t_] := \[Beta] Cos[\[Omega] t]
r[t_] := {s[t] + l Sin[\[Theta][t]], -l Cos[\[Theta][t]]}
v[t_] := r'[t]

Subscript[F, v][t_] := - \[Alpha] v[t]

L = 1/2 m Norm[v[t]]^2 + m g l Cos[\[Theta][t]]

Simplify[D[ D[ L, \[Theta]'[t] ], t] - D[L, \[Theta][t]] ==
Subscript[F, v][t] .
D[ r[t], \[Theta][t]] /. {Cos[\[Theta][t]] -> \[Theta][t],
Sin[\[Theta][t]] -> \[Theta][t],
Sin[\[Theta]'[t]] -> \[Theta]'[t],
Cos[\[Theta]'[t]] -> 1 - \[Theta]'[t]^2/2,
Cos[t \[Omega]] -> 1 - (t \[Omega])^2/2,
Sin[t \[Omega]] -> t\[Omega]}]
```

This yields scary equation

$$l(2\theta(t)\left(2gm+\beta\omega\left(mt^2\omega^3-2m\omega-2\alpha t\omega\right)\right)+4l\theta(t)^2(m\theta''(t)-$$
$$m\theta'(t)^2+2\alpha\theta'(t))+lm\left(\theta'(t)^4-8\theta'(t)^2+4\right)\theta''(t))=0$$

11/8/15

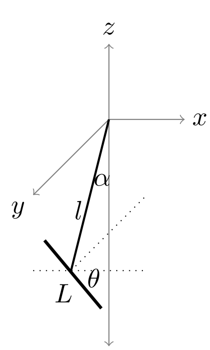
Correcting the viscous dampening force to only apply along $\dot{\theta}$ (and not the top pendulum displacement), by a near identical derivation, yields ODE

$$l\theta(t)\left(m\left(2g+\beta\omega^2\left(t^2\omega^2-2\right)\right)+4l\theta(t)\left(m\theta''(t)+\alpha\theta'(t)\right)\right)=0$$

13/8/15

Torsion pendulum.

Assume it receives all driving from the same direction ($\hat{y}$), at (causing whole pendulum swinging) and to the sides (causing rotation) of the suspension point (the center of mass).



We see that the range of z spanned by the pendulum body depends on both α and θ, as thus does the total gravitational potential energy in the Lagrangian formalism.

The suspension point is parameterized purely by α to be

$$\vec{s}(\alpha) = (0,\ l\sin(\alpha),\ -l\cos(\alpha)).$$

The pendulum body may then be directed in any direction along the plane orthogonal to $\vec{s}$.

Consider $\alpha = 0$, then our pendulum arm points (with arbitrary magnitude) along

$$\vec{p}(\theta, 0) = (\cos(\theta), \sin(\theta), 0)$$

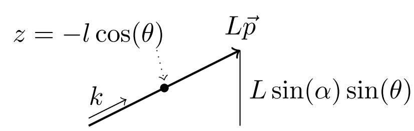
while the suspension point is $\vec{s}(0) = (0, 0, -l)$.

We can now consider increasing our α value as rotating our pendulum arm in 3D about the origin by angle α around the x axis. Since only the z values spanned by our pendulum arm matter, we merely rotate our pendulum arm by α and consider the new z-height of the suspension point; this will provide all necessary information to integrate along our pendulum arm for the total gravitational potential.

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

$$\begin{aligned} \vec{p}(\theta, \alpha) &= R_x(\alpha)\vec{p}(\theta, 0) \\ &= (\cos(\theta), \cos(\alpha)\sin(\theta), \sin(\alpha)\sin(\theta)). \end{aligned}$$

Notice $|\vec{p}(\theta, \alpha)| = 1 \ \ \forall(\theta, \alpha) \in \mathbb{R}^2$. This allows us to see instantly that at some θ, α, the suspension point is at z-height $-l\cos(\alpha)$ and the pendulum arm extends $\frac{L}{2}\sin(\alpha)\sin(\theta)$ above and below the suspension point in the z-dimension.



We observe that as k iterates from 0 to L across the pendulum arm, each point has z value

$$\begin{aligned} h(k) &= -l\cos(\theta) - \frac{L}{2}\sin(\alpha)\sin(\theta) + k\sin(\alpha)\sin(\theta) \\ &= -l\cos(\theta) + (k - \frac{L}{2})\sin(\alpha)\sin(\theta). \end{aligned}$$

The total gravitational potential energy of the bar is then

$$
\begin{aligned}
U(\theta, \alpha) &= \int \mathrm{d}m \; g \; h(k) \\
&= \int_0^L \frac{m}{L} g \left(-l\cos(\theta) + (k - \frac{L}{2}) \sin(\alpha)\sin(\theta) \right) \mathrm{d}k \\
&= -\frac{m}{L} gl\cos(\theta) L - \frac{L^2}{2}\frac{m}{L} g\sin(\alpha)\sin(\theta) + \frac{m}{L} g \sin(\alpha)\sin(\theta) \int_0^L k \; \mathrm{d}k \\
&= -mgl\cos(\theta) - \frac{Lm}{2} g\sin(\alpha)\sin(\theta) + \frac{m}{2} g\sin(\alpha)\sin(\theta) \\
&= -mgl\cos(\theta).
\end{aligned}
$$

Of course, I had missed an obvious property; that for a given α value, the total gravitational potential energy is constant despite any rotation in θ: when one side of the pendulum gains gravitational potential, the other symmetric side loses an equal amount.

For MAGPI, this symmetry doesn't exist but misalignment with the magnetic fields causes further potential terms.

Meeting 14/8/15

- Read through the MAGPI Mathematica notebook ☐
- Calculate the magnetic field of a 3rd year lab Helmholtz Coil ☑
- Estimate parameters of small-scale MAGPI setup ☑

17/8/15

* lots of progress-absent stuffing around with Mathematica *

19/8/15

Attempting to substitute the small-angle geometry of an oscillating pendulum suspension point into the existing ODE for non-damped, static suspension point motion...

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta = \frac{x_b - x_t}{l}$$

$$x_t = \alpha e^{i\omega t}$$

$$\ddot{x_b} + \alpha\omega^2 x_t = -w_0^2(x_b - x_t)$$

** Disregard: incorrectly substituting geometry of θ and suspension point and test mass displacements into the simple, undrive, undamped oscillator equation, which does not encapsulate the added dynamics on θ that the oscillating suspension point introduces. **

20/8/15

Helmholtz Coil Field

The center of the Helmholtz Coils have magnetic field

$$B = \frac{8}{5\sqrt{5}} \frac{\mu_0 n I}{R}$$

($\mu_0 = 4\pi \times 10^{-7}\,\mathrm{Tm/A}$) which points through the surfaces enclosed by the coils. Applying the variance formula, this has uncertainty

$$\delta B = \sqrt{\frac{64\delta\mathrm{R}^2 J^2 \mu_0^2 n^2}{125 R^4} + \frac{64\delta\mathrm{n}^2 J^2 \mu_0^2}{125 R^2} + \frac{64\delta\mathrm{J}^2 \mu_0^2 n^2}{125 R^2}}$$

Estimating the standard parameters of a Helmholtz Coil, we'll assume the loops are radius 10 cm ± 3 cm, which is also their seperation. We'll assume they have 150 ± 50 turns of wire around each loop, and can run currents through a range 10 to 70 Amperes, which we'll take as 40 ± 30 A.

This presents a magnetic field

$$0.05 \pm 0.05\,\mathrm{T}$$

with an actual value (maximum 0.1 T) depending on how the actual parameters of the coil turn out, in my estimated ranges. For future calculation, I'll assume we can achieve a horizontal magnetic field of strength 0.1 T at the suspension point with a typical laboratory Helmholtz Coil.

Consultation with Dr Russel Anderson

Whilst demonstrating the usage of Mathematica, Russel and I explored the homogeneity of the magnetic field of the Helmholtz Coil along the axis of separation of the coils.

Finding equilibrium between magnetic dipole and rod parameters (length and mass)

Let the rod have length L, with the magnetic and suspension point located at $x = 0$ along the rod ($x \in [0, L]$), and total mass m_{rod} distributed uniformly along x (for now, ignoring any test mass). Let the magnet have mass m_{magnet}.

For now, we'll assume the magnet is point-like and perfectly positioned at the suspension point so that it contributes no gravitational torque on the rod (but still contributes to the center of mass).

The center of mass mass of just the rod is intuitively at $x = \frac{L}{2}$ with a total mass m_{rod}, but the total center of mass is shifted by m_{magnet} at $x = 0$. Therefore

$$\begin{aligned} x_{\text{center of mass}} &= \frac{m_{\text{magnet}}(0) + m_{\text{rod}}(\frac{L}{2})}{m_{\text{magnet}} + m_{\text{rod}}} \\ &= \frac{L\ m_{\text{rod}}}{2(m_{\text{magnet}} + m_{\text{rod}})} \end{aligned}$$

At this point, gravity applies force (downward; orthogonal to the assumed always horizontal rod)

$$F_g = -(m_{\text{magnet}} + m_{\text{rod}})g$$

and therefore applies torque

$$\begin{aligned} \tau_g &= F_g x_{\text{center of mass}} \\ &= -\frac{gL}{2} m_{\text{rod}}. \end{aligned}$$

This torque must be combat entirely by the magnetic interaction between the magnet of dipole magnitude μ and the always orthogonal magnetic field. Assuming perfect alignments of everything,

$$\begin{aligned} \tau_{\text{magnetic}} &= -\tau_g \\ \mu &= \frac{gL}{2B} m_{\text{rod}}. \end{aligned}$$

The strength of the magnetic moment required is inversely proportional to the field our Helmholtz Coil can produce. We'll now hope to find the parameters (such as mass and size) of a magnet which can support such a moment.

The paper mentions...
"Ferrimagnetic material such as YIG has both a large remanence and a low conductivity, meaning that it can support a relatively large magnetic field while suppressing eddy currents [12]. If we assume that the permanent magnet is composed of YIG with 1 mm-thick lamination, the eddy current noise can be limited to... which is below the existing limiting noise sources"

short research on ferrimagnetic material

The Curie temperature (above which, magnetic properties of the material are only observed under magnetic induction) of Yttrium Iron Garnet is 560 K, which is safely high for our purposes. Finding the magnetic moment of a sample of YIG of any geometry and mass is difficult!

Magnetic moment required to suspend a length L, mass m_{rod} aluminium rod with no test mass in a Helmholtz Coil

Let's assume our rod is aluminium and cylindrical with a small radius 0.5 cm and length 100 cm. It thus has volume $100\pi 0.5^2\,\text{cm}^3 \approx 80\,\text{cm}^3$ and a mass (according to http://www.coolmagnetman.com/magconda.htm) of $80 \times 2.7g \approx 0.2\,\text{kg}$.

At maximum environmental field magnitude $B \approx 0.1\,\text{T}$ (for the Helmholtz Coil of modest parameters; 200 turns, 13 cm seperation / radius, 70 A current), the magnetic moment of our magnet required for suspension is

$$\begin{aligned}\mu &\approx \frac{9.8}{2 \cdot 0.1} 0.2 \text{ N m/T} \\ &= 9.8 \text{ N m/T}\end{aligned}$$

Though actually depending on the homogeneity of the Helmholtz Coil magnetic field off the axis of separation of the coils (which I'm yet to calculate), the limit of the vertical height (the length along the dipole axis) is probably on the scale of cm.

Since good rare-earth metal magnets have typical field strengths of around 1 T, achieving such a magnetic moment as required would necessitate (by $\mu = pl$) magnets of height on the order of 10's of meters. This is extremely unsuitable, and this doesn't yet even consider the weight of the test mass!!

I've learned that the magnetic forces aren't so strong as to be careless with the other parameters of the rod. I'll now look at tweaking all parameters (in Mathematica) to find a configuration which results in a smaller magnetic moment, around 0.1 Nm/T.

Relationship between parameters required to balance rod torques (with test mass, but no sweet spot)

Please refer to the Mathematica document ***20-8-15 small scale MAGPI parameters*** for a derivation of the following.

For a rod of mass m_{rod}, length l_{rod} with a test mass of mass m_{test} fixed at distance $0 < x_{\text{test}} < l_{\text{rod}}$ from the *left* side of the rod, where the left end of the rod features a fixed magnet of moment μ (of arbitrary mass), centered inside the magnetic field of a Helmholtz coil featuring loops of radius / separation R, a number n turns each with a current of J supplied through them, the required magnetic moment of the magnet to sustain the rod perfectly parallel is

$$\mu = \frac{5\sqrt{5} g R (l_{\text{rod}} m_{\text{rod}} + 2 m_{\text{test}} x_{\text{test}})}{16 J n \mu_0}.$$

The most direct constraints are on μ, the magnetic moment of the magnet, which may vary linearly with the height of the magnetic; the limits thereon will require investigation the homogeneity of the Helmholtz Coil's magnetic field off its axis of seperation.

For now, since many *good* rare-earth-metal magnets can achieve field strengths of 1 T and moments proportional to their length (which we'll permit to be 3 cm for now) of 0.03 mN/T, we'll substitute this and the other most difficult to modulate parameters, then solve for the remaining.

The parameters assumed (the first two being unvariable constants) are $g = 9.8\,\text{m/s}$, $\mu_0 = 4\pi 10^{-7}\,\text{Tm/A}$, $\mu = 0.03\,\text{mN/T}$, $J = 40\,\text{A}$, $n = 200$, $R = 5\,\text{cm}$ and importantly, that the test mass is fixed at the very right-most end of the rod (at $x = l_{\text{rod}}$). This leaves the relationship

$$l_{\text{rod}}(m_{\text{rod}} + 2m_{\text{test}}) \approx 0.0009\,\text{kg m}.$$

We can see already this is incredibly hard to achieve. If even the mass of the rod and test mass combined were 1 kg, the rod would require a length of at maximum $0.0009\,\text{m} \approx 0.9\,\text{mm}$, which is ridiculously too small!

To balance torques on a longer rod, our assumed parameters should change. We'd require

- a decrease in the Helmholtz Coil loop radius R from 5 cm.
- an increase in the current through the Helmholtz Coil J from 40 A.
- an increase of the number of turns of wire in the loops of the Helmholtz Coil from 200.

Increasing these will allow a bigger $l_{\text{rod}}m_{\text{rod}}+2m_{\text{test}}x_{\text{test}}$ term that still leaves the magnetic moment of the fixed magnet small. However, the actual length of the rod l_{rod} must be orders of magnitude larger than that previously calculated permitted by our assumed parameters, and the above parameters can not afford to increase so significantly.

Help!

Relevant mathematica code

Listing 4: Calculating magnetic field of Helmholtz Coils

```

The magnetic field at the center of a Helmholtz Coil (with seperation between loops equal to the radius of the
    loops) is ...

B[n_, I_, R_] := 8/(5 Sqrt[5]) Subscript[\[Mu], 0] n I/R

with uncertainty (by the Variance formula, or from ' first principles with quadrature')

\[Delta]B[n_, \[Delta]n_, I_, \[Delta]I_, R_, \[Delta]R_] := \[Sqrt]((D[B[n, I, R], n] \[Delta]n)^2 +
(D[ B[n, I, R], I] \[Delta]I)^2 + (D[B[n, I, R], R] \[Delta]R)^2)

If we assume the Helmholtz loops have radius 10cm \[PlusMinus] 3cm, 150 \[PlusMinus] 50 turns and take current
    40\[PlusMinus]30A, then we can produce magnetic fields

B[n, J, R] /. {n -> 150, J -> 40, R -> 0.1, Subscript[\[Mu], 0] -> 4 \[Pi] 10^-7}
= 0.0539506

\[Delta]B[n, \[Delta]n, J, \[Delta]J, R, \[Delta]R] /. {n -> 150, \[Delta]n -> 50, J -> 40, \[Delta]J -> 30,
    R -> 0.1, \[Delta]R -> 0.03, Subscript[\[Mu], 0] -> 4 \[Pi] 10^-7}
= 0.0471446
```

Listing 5: Balancing torques on the MAGPI rod in generality

```
mrod    mass of the rod itself
lrod    total length of the rod
mtest   mass of the test mass
xtest   position of the test mass (0 < xtest < L)
\[Mu]   magnetic moment of the magnet (fixed to left end of rod)
mmag    mass of the magnet

xcen    center of mass of the total system (ignoring suspension wire)

n    number of turns in the loops of the Helmholtz Coil
J    current through the loops of the Helmholtz Coil
R    radius of the loops and their seperation in the Helmholtz Coil
\[Mu]0    permeability of free space (4 \[Pi] 10^-7 Tm/A)

\[Tau]g   the torque applied on the center of mass by gravity
\[Tau]m   the torque applied to the magnet by the magnetic field *

The magnetic field at the center of the Helmholtz Coil is

B = \[Mu]0 8/(5 Sqrt[5]) n J/R;

The center of mass of the total system (ignoring suspension wire) is
```

```

xcen = (xtest mtest + mrod lrod/2)/(mrod + mtest + mmag);

The gravitational torque on the total system is then

\[Tau]g = − g (mrod + mtest + mmag) xcen ;

The magnetic torque on the magnet and thus system is

\[Tau]m = \[Mu] B;

Balancing the torques presents the equation

torquebalance = (−\[Tau]g == \[Tau]m);

which requires our magnet to have magnetic moment

Solve[torquebalance, \[Mu]]

{{\[Mu] −> (5 Sqrt[5] g R (lrod mrod + 2 mtest xtest))/(
16 J n \[Mu]0)}}

Solve[torquebalance, \[Mu]] /. {g −> 9.8, \[Mu] −> 0.03, \[Mu]0 −> 4 \[Pi] 10^−7, J −> 40, R −> 0.05, n −>
    200, xtest −> lrod}
{{0.03 −> 34.0589 (lrod mrod + 2 lrod mtest)}}
```

Arbitrary thoughts

- Explore the homogeneity of the magnetic field off-axis (around the center of the axis of separation) and compare to the dimensions of the magnetic along its dipole axis.

- Can we use something other than the helmholtz coil to make a shaped magnetic field which will result in smaller amplitude precessions of the magnet? Some parabeloid shaped field which will help the suspension wire's dampening of the magnetc procession? Is there a geometry to do this without breaking the homogeneity along the axis of the pendulum?

Meeting 21/8/15

- – Explore magnetic field homogeneity of finite solenoid, spherical solenoid, Maxwell-helmholtz coils ☐
- – Formulate magnetic moment in terms of magnet dimensions ☐
- – Study the *Fluctuation Dissipation Theorem* ☑
- – Configure remaining small-scale MAGPI setup parameters by optimising sweet spot ☐

21/8/15

Consultation with Dr Eric Thrane

Eric and I discussed the physics and motivation behind the *sweet spot*. At the center of mass, the seismic frequency response of the torsion pendulum decays exponentially* for frequencies above the critical.

At positions beyond the center of mass, the gravitational transfer function retains unity* whilst the seismic frequency response shows a sharp dip to zero (theoretically) at a frequency proportional to the displacement of the position from the center of mass. This means that a particular seismic frequency can be almost completely attenuated by careful choice of position past the center of mass at which to listen.

However, the MAGPI team have chosen to push the attenuated frequency to $f \to \infty$ (which, by the asymptotic behaviour in the relationship between distance from center of mass against frequency, results in a finite distance) for several reasons. Firstly, exponential decay of seismic response above the critical frequency is already sufficient attenuation, and seismic suppression is less important than g-coupling. Secondly, the transfer function is augmented around the dip at the chosen frequency, so results in greater response of the pendulum to seismic noise at neighbouring frequencies anyway. Thirdly, the microseism is not localized to any particular frequency, but yields seismic noise across a large bandwidth, so there is no appreciable benefit to attentuating a particular frequency.

Essentially (as far as I understand), the attenuation of particular seismic frequency holds no benefit, so the *sweet spot* is chosen to push the attenuation to $f \to \infty$ so as not to disturb the exponential form of the transfer function.

What happens AT the center of mass?

27/8/15

good page on torsion pendulums

http://www.physics.csbsju.edu/~jcrumley/370/tp/torsion_pendulum.html

Fluctuation dissipation theorem

The fluctuation dissipation theorem draws a parallel between the behaviour of the system at thermal equilibrium and the system's response (as an equilibrium restoration) to small, externally-caused pertubations from thermal equilibrium. That is, as well put here http://www.mrc-lmb.cam.ac.uk/genomes/madanm/balaji/kubo.pdf, *"[the] theorem states a general relationship between the response of a given system to an external disturbance and the internal fluctuation of the system in the absence of the disturbance."*

That is to say if the system shows some characteristic behaviour when it is displaced from thermal equilibrium (say by some small driving force, which is assumed weak enough so as not to increase the rates of relaxation of the system to equilibrium), then the same behaviour will be observed due to random thermal fluctuations of the undisturbed system about equilibrium.

Magnetic moment

Good breakdown of types of magnetism: http://www.irm.umn.edu/hg2m/hg2m_b/hg2m_b.html

Magnetic moment of a solenoid of N turns carrying current I and vector area $\vec{S}$:

$$\vec{\mu} = NI\vec{S}.$$

Magnetic moment of a closed current loop:

$$|\vec{\mu}| = I \times \text{Area}.$$

(similar formulation for arbitrary current densities)

A magnetic moment in an external magnetic field has potential energy

$$U = -\vec{\mu} \cdot \vec{B}.$$

Configuring remaining small-scale MAGPI parameters

by conservative rod length

These calculations are wrong: the magnetic moment of 2000 is achievable with a significantly larger (around 100 times in volume) YIG magnet.

By assuming standard parameters of the Helmholtz Coil ($I = 40\,\text{A}$, $R = 5\,\text{cm}$, $n = 200$) we were left with balance equation

$$\mu \approx 34.0589 \left(l_{\text{rod}} m_{\text{rod}} + 2 x_{\text{test}} m_{\text{test}} \right).$$

Though I've been unsuccessful in finding a formulation of the magnet moment of a ferrimagnet of a particular geometry and remanent magnetization, if we assume a more sensible moment of $\mu \approx 2000\,\text{J/T}$, our constraint becomes

$$59 \approx l_{\text{rod}} m_{\text{rod}} + 2 x_{\text{test}} m_{\text{test}}$$

and we impose a conveniant rod length of 10 cm, which I'll assume cylindrical of radius 5 mm and to have a similar mass/volume ratio as Aluminium, then we have a total rod volume of $V = 0.1 \cdot (\pi 0.005^2) \approx 7.9 \times 10^{-6}\,\text{m}^3$ with total mass $m_{\text{rod}} = 2720V \approx 0.021\,\text{kg}$ and our constraint becomes

$$30 \approx x_{\text{test}} m_{\text{test}}.$$

Since the test mass must be located somewhere along the length of the rod ($0\,\text{m} < x_{\text{test}} < 0.1\,\text{m}$), we can support a minimum (if we positioned it right at the end of the rod, maximising its torque) of 300 kg, which is already *more than enough*.

by sweet-spot optimisation

Having demonstrated that a conservative rod length is easily supportable, we'll now explore the balance between rod length, position of the test mass and its mass, based on positioning the test mass such that the sweet spot is at the exact end of the rod.

Meeting 28/8/15

- Make a Mathematica notebook to propogate small-scale MAGPI parameter changes to system ☑
- Additionally consider affect on sweet-spot, the energy dissipation required by the coils, etc. ☑

28/8/15

Borrowed Dr Lincoln Turner's copy of *Classical Electrodynamic, Second Edition, J.D.Jackson.*

29/8/15

Begun formatting the Mathematica notebook.

Someone's derivation of Helmholtz fields http://jholzgrafe.com/wp-content/uploads/2012/12/Inductor_Lab.pdf

Finding magnetic moment of ferrimagnet - Attempt #1

I hope to find the proportionality between the magnitude of magnetic moment μ and the product of remanent magnetization and volume of the ferrimagnetic BV. I suspect the coefficient depends on the geometry of the magnet, and μ_0.

According to Classical Dynamics (J.D. Jackson, 2nd Edition, page 195), the magnetization of a spherical magnet of radius a with inner magnetic field B_{in} is

$$M = \frac{3}{8\pi} B_{\text{in}}$$

whilst the magnetic moment is

$$\begin{aligned} m &= \frac{4\pi a^3}{3} M \\ &= \frac{a^3}{2} B_{\text{in}}. \end{aligned}$$

We note this is in Gaussian units; an equivalent formula in SI units (where both m and B_{in} become SI) is therefore is

$$m = \sqrt{\frac{\mu_0}{4\pi}} \frac{a^3}{2} 10^4 B_{\text{in}}.$$

If we assumed this was of the form $m = kVB_{\text{in}}$ where $V = \frac{4}{3}\pi a^3$, then the constant of proportionality k is

$$\begin{aligned} k\frac{4}{3}\pi a^3 &= 10^4 \sqrt{\frac{\mu_0}{4\pi}} \frac{a^3}{2}, \\ \therefore k &= \frac{3}{8\pi} \sqrt{\frac{\mu_0}{4\pi}} 10^4 \\ &= \frac{3\sqrt{10}}{8\pi} \end{aligned}$$

Since we'd like to express this in terms of the permeability of free space $\mu_0 = 4\pi 10^{-7}$ (which usually appears in the denominator), we'll express k as

$$\begin{aligned} k &= \frac{1}{\mu_0} \left(\frac{3\sqrt{10}}{8\pi} \right) \left(4\pi 10^{-7} \right) \\ &= \frac{3\sqrt{10}}{2} \frac{1}{\mu_0} \cdot 10^{-7}. \end{aligned}$$

We've shown that the magnetic moment (at least for a sphere) satisfies

$$m = \frac{3\sqrt{10}}{2} 10^{-7} \frac{V B_{\text{in}}}{\mu_0}.$$

This seems concerningly small! We'll test it with our cylindrical magnet of volume $V = 0.04 \times (\pi 0.04^2)\,\text{m}^3 \approx 2 \cdot 10^{-4}\,\text{m}^3$ of magnetic field strength $\approx 2\,\text{T}$ with an expected magnetic moment of $\approx 20\,\text{J/T}$. Our formula suggests

$$m = \frac{3\sqrt{10}}{8\pi}(2)(2 \cdot 10^{-4}) \approx 0.00015.$$

We immediately see our calculated magnitude is 5 orders of magnitude wrong. Back to the drawing boards!

3/9/15

Continued work on small-scale MAGPI parameter Mathematica notebook (restructured, added thermal physics)

Heat Generation in the Helmholtz Coil

Is the power consumed by the coil all invested into heat? Does the maintenance of the magnetic field draw power? In a static system, the field doesn't perform any work and so should consume no power, suggesting all power drawn by the helmholtz coil is converted to heat.

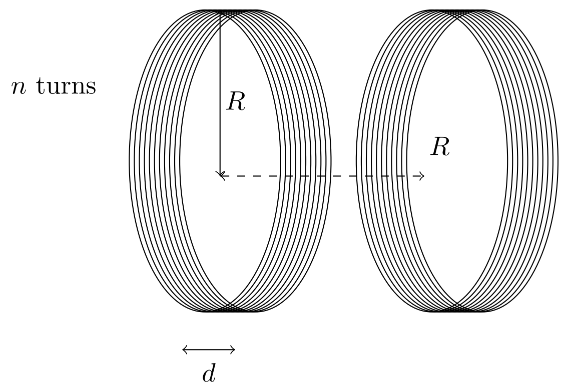
Assuming so, let Res be the total resistance of the coils in the Helmholtz Coils. We rearrange the equation of the field strength produced by the coils to find the power generated (assumably as heat) as a function of the Helmholtz parameters the produced field.

$$\begin{aligned} P &= J^2 Res \\ &= \left(\frac{RB}{\mu_0 n} \frac{5\sqrt{5}}{8} \right)^2 Res \end{aligned}$$

This power should also feature radiation absorbed from the environment

Resistance of the loops

http://www.coilgun.eclipse.co.uk/coil_resistance_formula.html



The total length of wire between both coils is

$$l = 4\pi R n.$$

We'll use copper wires with a resistivity of $\rho \approx 1.7 \times 10^{-8}$. I'm going to assume the copper wire is similar in size to the nib of my pen, with a radius of $r = 1\,\text{mm}$ and therefore a cross-sectional area of

$$A = \pi r^2.$$

The total resistance of both (combined) loops is then

$$\begin{aligned} Res &= \rho \frac{l}{A} \\ &= \rho \frac{4Rn}{r^2}. \end{aligned}$$

Thermal properties of the loops (modelled as square cross-section)

This suggests a total heat generation of

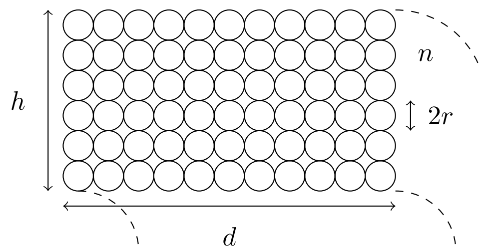
$$\begin{aligned} P &= \left(\frac{RB}{\mu_0 n} \frac{5\sqrt{5}}{8} \right)^2 \rho \frac{4Rn}{r^2} \\ &= \frac{125}{16} \frac{R^3 B^2 \rho}{\mu_0 n r^2}. \end{aligned}$$

We'll seek to compare this to the Stefan Boltzmann law which says our power radiated is

$$P = \epsilon \sigma A T^4, \qquad \sigma \approx 5.67 \times 10^{-8}\,\text{Wm}^{-2}\text{K}^{-4}.$$

By 'painting our wires black', our coils become perfect absorbers (and emitters) resulting in an emissivity of $\epsilon \approx 1$. We now need to find the surface area of our loops, noting it smaller than the total surface area of our wires; our wires are arranged in bundles within which any emitted radiation is immediately absorbed by another wire. We seek to find the surface area of the *cover* of this shape.

Each loop features n turns of wire, each of diameter $2r$ which are arranged to form a loop width of d. Let h be the "height" of the loop cross-section.



A width d can fit $\frac{d}{2r}$ wires horizontally (we'll assume this produces an integer number), suggesting a height (dimensionless; as a number of wires)

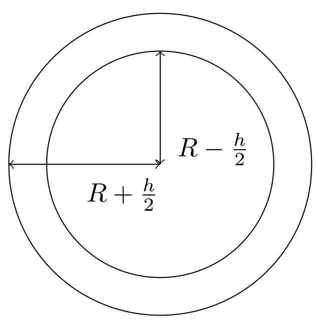
$$h^* = \frac{n}{\left(\frac{d}{2r}\right)} = \frac{2nr}{d}.$$

As a length

$$\begin{aligned} h &= h^*(2r) \\ &= \frac{4r^2 n}{d}. \end{aligned}$$

Ideally, we'd want $h < d$ so that our formulation of the produced magnetic field (which is a function of R; the radius of the assumed infinitesimally thin wire surface) isn't invalidated.

For simplicity, we'll take R to be the radius of 'middle' layer of wires. Both sides of each loop than appear as



each with area

$$\begin{aligned} A_{\text{side}} &= \pi(R + \frac{h}{2})^2 - \pi(R - \frac{h}{2})^2 \\ &= \pi(R + \frac{2nr^2}{d})^2 - \pi(R - \frac{2nr^2}{d})^2 \\ &= \frac{8\pi n R r^2}{d}. \end{aligned}$$

The inner face of each loop has surface area of a cylinder of radius $R - \frac{2nr^2}{d}$ and length d:

$$\begin{aligned} A_{\text{inner}} &= 2\pi(R - \frac{2nr^2}{d})d \\ &= 2\pi(Rd - 2nr^2). \end{aligned}$$

The outer face of each loop is similarly

$$\begin{aligned} A_{\text{outer}} &= 2\pi(R + \frac{2nr^2}{d})d \\ &= 2\pi(Rd + 2nr^2). \end{aligned}$$

Both loops combined (which between them, feature 4 side faces, 2 inner faces and 2 outer faces) have surface area

$$\begin{aligned} A &= 4A_{\text{side}} + 2A_{\text{inner}} + 2A_{\text{outer}} \\ &= \frac{32\pi nRr^2}{d} + 4\pi(Rd - 2nr^2) + 4\pi(Rd + 2nr^2) \\ &= \frac{32\pi nRr^2}{d} + 8\pi Rd. \end{aligned}$$

We can now plug this into the Stefan-Boltzmann law to find the power radiated as a function of temperature of the wires.

$$\begin{aligned} P &= \sigma A T^4 \\ &= \sigma \left(\frac{16\pi nRr}{d} + 8\pi Rd \right) T^4 \end{aligned}$$

Why this model is terrible: assumes the wires a perfectly grid-like (in their cross-section) when in reality, the wires would fill the grooves and be much more dense. Note however that remodelling is not necessary; we're interested in the surface area only to find the temperature at which all generated heat can be radiated, which varies with surface area as $O(\frac{1}{A^{1/4}}$. The temperature is very insensitive to changes in area, so our approximate model is sufficient.

Finding the equilibrium temperature

(Just a function of the other Helmholtz parameters; established in Mathematica)

Finding magnetic moment of ferrimagnet - Attempt #2

We employ the conversions between units as outlined on page 819 of Jackson, also consulting `http://uspas.fnal.gov/materials/02Yale/Units.pdf`.

Point of confusion: some sources say magnetic fields ('Magnetic induction') are converted by a factor of 10^4 (e.g. the url above) whilst others say $\sqrt{\frac{4\pi}{\mu_0}} = 10^{7/2} \neq 10^4$ (e.g. Jackson). What am I missing??! Below, we'll trust Jackson.

For the spherical magnet of radius a,

$$\vec{B}_{\text{gaussian}} = \frac{8\pi}{3}\vec{M}_{\text{gaussian}}$$

and

$$\begin{aligned}\vec{\mu}_{\text{gaussian}} &= \frac{4\pi a^3}{3}\vec{M}_{\text{gaussian}} \\ &= \frac{4\pi a^3}{3}\frac{3}{8\pi}\vec{B}_{\text{gaussian}} \\ &= \frac{a^3}{2}\vec{B}_{\text{gaussian}}.\end{aligned}$$

We now substitute that $B_{\text{gaussian}} = \sqrt{\frac{4\pi}{\mu_0}}B_{\text{MKS}}$ and $\mu_{\text{gaussian}} = \sqrt{\frac{\mu_0}{4\pi}}\mu_{\text{MKS}}$ to find

$$\begin{aligned}\vec{\mu}_{\text{MKS}} &= \sqrt{\frac{4\pi}{\mu_0}}\frac{a^3}{2}\sqrt{\frac{4\pi}{\mu_0}}\vec{B}_{\text{MKS}} \\ &= \frac{2\pi a^3}{\mu_0}\vec{B}_{\text{MKS}}.\end{aligned}$$

We hypothesize that in general, the magnetic moment is proportional to the volume of the magnet and some geometry based coefficient. We therefore substitute $V_{\text{sphere}} = \frac{4}{3}\pi a^3$.

$$\vec{\mu}_{\text{MKS}} = \frac{3}{2}\frac{V_{\text{sphere}}\vec{B}_{\text{MKS}}}{\mu_0}.$$

Unable to find a strong relation between $\frac{3}{2}$ and a sphere, we tentatively assume this is the coefficient for any ferrimanget (of at least a reasonable similarity to a sphere).

Our formula suggests that for our cylindrical magnet of volume $V \approx 6 \cdot 10^{-5}\,\text{m}^3$ and approximate remanent field strength $B \approx 2\,\text{T}$ (with expected moment $\approx 20\,\text{J/T}$) has magnetic moment

$$\begin{aligned}\mu_{\text{MKS}} &= \frac{3}{2}\frac{6\times 10^{-5}\cdot 2}{4\pi 10^{-7}}\text{ J/T} \\ &\approx 50\text{ J/T}\end{aligned}$$

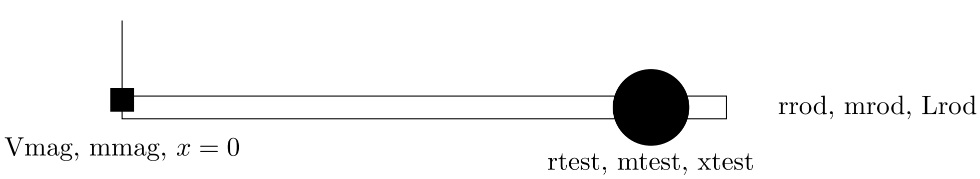
*********************************8

This is an order of magnitude larger than expected. Where now lies the problem?!. It's not the $\sqrt{\frac{4\pi}{\mu_0}} = 10^{7/2} \neq 10^4$ mystery, is it?!

More food for thought: http://www.wired.com/2014/01/measuring-strength-magnet/. This site claim the magnetic field due to a dipole μ is $B = \frac{\mu_0\mu}{4\pi}\frac{1}{r^3}$ where r is the distance from the center of the dipole. Confirmed here: http://www2.vernier.com/sample_labs/PWV-31-COMP-magnetic_field_permanent_magnet.pdf.

4/9/15

Moment of Inertia (of the rod about the suspension point)



Since moment of inertia is additive, we can simply consider the moment of inertia of each component in our system about the suspension point and sum them to find the total moment. We find them by a table of moments of inertias and by the parallel axis theorem.

Moment of the rod

Assuming the rod to be infinitely thin (but rigid), then its moment of inertia has the form of a thin rod being rotated at an end point;

$$I_{\text{rod}} = \frac{m_{\text{rod}} L_{\text{rod}}^2}{3}.$$

Moment of the test mass

If rotated about its own center, the spherical test mass has moment of inertia

$$I^* = \frac{2 m_{\text{test}} r_{\text{test}}^2}{5}.$$

We apply the parallel axis theorem to find its moment of inertia about the suspension point:

$$\begin{aligned} I_{\text{test}} &= I^* + m_{\text{test}} x_{test}^2 \\ &= \frac{2 m_{\text{test}} r_{\text{test}}^2}{5} + m_{\text{test}} x_{test}^2. \end{aligned}$$

Moment of the ferrimagnet

We model the ferrimagnet as a solid cylinder of radius r_{mag} and height h_{mag} with cylindrical axis aligned with the suspending line.

$$I_{\text{mag}} = \frac{m_{\text{mag}} r_{\text{mag}}^2}{2}.$$

CORRECTION 24/9/15: We'll instead use a spherical suspended magnet. Therefore its MOI about its center is

$$I_{\text{mag}} = \frac{2}{5} m_{\text{mag}} r_{\text{mag}}^2$$

Total moment

The total moment of inertia about the suspension point is then

$$\begin{aligned} I &= I_{\text{rod}} + I_{\text{test}} + I_{\text{mag}} \\ &= \frac{m_{\text{rod}} L_{\text{rod}}^2}{3} + \frac{2 m_{\text{test}} r_{\text{test}}^2}{5} + m_{\text{test}} x_{test}^2 + \frac{2}{5} m_{\text{mag}} r_{\text{mag}}^2. \end{aligned}$$

Mathematica parameters

Completed the first iteration of the small-scale MAGPI parameter MMA notebook. The current issues requiring address are

- The radius of the pendulum rod is not yet incorporated into the moment of inertia about the suspension point (so far, the rod is assumed infinitely thin). Integrate!
- The rate of heat generation in the helmholtz coils is an underestimate since it does not consider absorption from environment. Ask Dr Turner!
- The magnetic moment of the ferrimagnet is hard coded; as seen somewhere above, the derivation of its formula in terms of magnet geometry has been difficult.
- The formulation of the sweet spot may be wrong or inapplicable to small-scale. Is $\kappa = \frac{MR^2}{I}$ in the paper assuming $m_{\text{test}} >>$ the rest of the system?
- The calculated surface area of the Helmholtz Coils are unsettingly large.

Disregard!

Some errors in my calculations were identified, resolving the unusually large surface area but introducing new physical complications. Note that environmental thermal *gain* of the Helmholtz coil is disregarded (or at least negligable) since the coil will always be at a larger temperature than its environment (i.e. Newton's law of cooling).

The current issues are now

- The calculated temperature of thermal equilibrium (at which, the coil can radiate away all electrically generated heat) is significantly high; $\approx 700\,\text{K}$. This is certainly high enough to damage the coil and setup.
- The radius of the pendulum rod is not yet incorporated into the moment of inertia about the suspension point (so far, the rod is assumed infinitely thin). Integrate!
- The magnetic moment of the ferrimagnet is hard coded; as seen somewhere above, the derivation of its formula in terms of magnet geometry has been difficult.
- The formulation of the sweet spot may be wrong or inapplicable to small-scale. Is $\kappa = \frac{MR^2}{I}$ in the paper assuming $m_{\text{test}} >>$ the rest of the system?

Meeting 4/9/15

- Model a setup where heat is evacuated by a copper heat sink and ice bath ☑

Rejection of radiation reliant cooling

Running 25A through a Helmholtz coil of 200 turns of radius 1mm (black; emissivity 1) copper wire wrapped around width 2cm, radius 5cm cylinder frames produces a sufficient torque on a 300g magnet of moment 20J/T to suspend a 4g, 20cm rod with an 800g test mass located 18cm along it with a sweet-spot at 19cm.

However, the Helmholtz coil draws 425W, requiring a temperature of 560K to evacuate by radiation, one which is largely insensitive to changes in power and surface area of the coils.

7/9/15

Corrected a miscalculation of the sweet spot in the small-scale MAGPI parameter MMA notebook, which used the moment of inertia about the suspension point instead of about the center of mass as the paper employs.

This seems to now result in a sweet spot position that is *left* of the center of mass (to be be between the suspension point and the center of mass), which is surprising and concerning. Is this physically acceptable?

10/9/15

We now model the evacuation of heat from the Helmholtz coils to a copper enclosing.

I assume that the thermal diffusivity is significant so that the rate of thermal conduction through the enclosing to the ice bath is significantly large, allowing me to assume that input heat to the enclosing is instantaneously uniformly distributed through its volume.

[Working in physical exercise book]

Meeting 11/9/15

– Remodel heat conduction between coils and copper enclosing, and heat conduction through copper to bath ☑

– Explore water-cooled Helmholtz coils and the resulting vibrational pertubation of magnetic field around ferromagnet (tilting) ☐

– Characterize the vibrational spectrum of the coils (use transfer functions) ☐

– Begin compiling designs/results in formal report (Contact Professor Helmerson for example report) ☑

– Explore external field generation by permanent (YIG) ferrimagnets. ☑

– Create a git repository for logbook, report and MMA notebooks. ☑

– Calculate the suspension wire thickness required to achieve resonant frequency of the small-scale MAGPI of 1000s ☐

14/9/15

Created private GIT repo on GitLab.com

15/9/15

- created and formatted Report document (in LaTeX)
- first drafted the abstract

- created MMA notebook with sections/calculations parallel to Report sections
- Formally presented Heat Evacuation by Conduction Helmholtz design in Report

16/9/15

- created a private repo on GitLab with first commit
- added 2 transfer function derivations to Report notebook
- developed estimations for maximum achievable rate of heat flow for Copper (which is an electrical conductor, permitting eddy currents and by the fluctuation dissipation theorem, disturbing the external magnetic field aorund the pendulum! Oops! Consider instead an electrically insulating, thermally conducting cermamic such as Silicon Carbide).

17/9/15

- Deleted GitLab repo, instead moved to hgweb @ monash (must learn Mercurial :c)

SECTION REMOVED FROM REPORT

Reasons:

- Copper is a poor choice of enclosing material (permits eddy currents from external magnetic field which by the dissipation fluctuation theorem, will perturb the magnetic field)
- Ice-bath constant temperature is safe and doesn't warrant calculation (or at least, its own section. Appendix a brief version?).
- Wire-enclosing interface area is much more critical than through-enclosing interface rate*** (verify) so cylindrical modelling is silly.

Consider some arbitrarily shaped, highly thermally conductive (here, Copper) enclosing around the loops of the Helmholtz Coil, which join and are connected to an icebath. Good thermal and poor electrical conduction between the enclosing and the Helmholtz loops can be simultaneously achieved by wrapping the loops in some highly thermally conductive insulator, such as Ask Russel! and maximising contact this insulator and the enclosing. This insulating material can withstand temperatures $\approx 100\,^{\circ}\mathrm{C}$, which we'll take as a constraint on the maximum temperature of the coils and copper at which thermal equilibrium may be achieved.

The expected critical feature of this model of heat evacuation is the relationship between the rate of heat flow through the enclosing and its geometry, and whether a sufficient rate can be realistically achieved to conduct the notably high power dissipated by the Helmholtz Coils. To simplify our analysis, we can consider the ice-bath to be at a constant temperature of $0\,^{\circ}\mathrm{C}$; a reasonable assumption as motivated below.

Motivating a constant ice-bath temperature assumption

Without assuming any geometry of the enclosing, we can calculate a realistic, strict *lower* bound on the time taken for the icebath to rise from its initial $0\,^{\circ}\mathrm{C}$ to room temperature ($20\,^{\circ}\mathrm{C}$), by assuming the enclosing to have an infinite thermal conductivity factor $\kappa \to \infty$ such that all heat is transmitted from the loops to the ice bath instantaneously. We will also assume all ice in the bath has already melted to water so that no additional energy is consumed through phrase transition, further decreasing our calculated lower bound.

Since across $T \in [0, 20]\,^\circ\mathrm{C}$, the density of water varies by a negliable 0.2% from $\rho \approx 10^3\,\mathrm{kg\,m^{-3}}$ [**water˙density**] and its heat capacity at 1 atm by 0.8% from $C = 4.2 \times 10^3\,\mathrm{J\,kg^{-1}\,K^{-1}}$ [**water˙heat˙capacity**], we'll assume these quantities are constant. If we consider a filled ice bath of typical volume 40 L and thus mass $m = 40\,\mathrm{kg}$, the total energy required input to the bath is then

$$P\Delta t = Cm\Delta T,$$

which for the power $P \approx 425\,\mathrm{W}$ supplied by our Helmholtz Coils, suggests a duration of $\Delta t \approx 2.2\,\mathrm{h}$ to raise the temperature by $\Delta T = +20\,^\circ\mathrm{C}$.

This calculation ignores thermal gain by the bath from the environment, achievable by thermal insulation of the bucket.

None of this is necessary: it's incredibly easy to keep the ice bath at $0\,^\circ\mathrm{C}$. The input power is almost a sixth of a typical electric kettle power whilst the volume of water is more than 10 times larger. Since the phase transition from ice to water will consume energy whilst fixing the temperature at $0\,^\circ\mathrm{C}$, the bath will physically stay at $0\,^\circ\mathrm{C}$ until (essentially, assuming uniform heat distribution) all ice has melted.

Since even this conservative lower bound on the time available before the ice bath warms to room temperature (at which heat evacuation does *not* necessarily stop) is ample time for experimentation (typical oscillations are of the order 15 minutes? Ask Dr Thrane!), we can safely assume the ice bath maintains a constant temperature of $0\,^\circ\mathrm{C}$ in our analysis of the enclosing's rate of heat transfer.

Calculating the maximum distance from coils to bath for full heat transfer

Fourier's law presents the one-dimensional rate of heat flow P through a homogeneous material of thermal conductivity κ and cross-sectional area A (orthogonal to the temperature gradient $\frac{\Delta T}{\Delta x}$) as

$$P = -\kappa A \frac{\Delta T}{\Delta x}. \qquad \text{[\textbf{fouriers˙law}]}$$

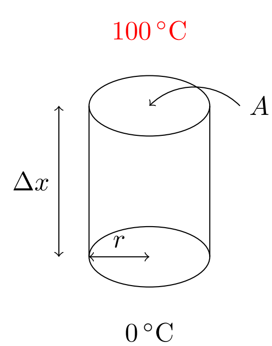


Figure 1: Depiction of a cylindrical Copper connection between the Helmholtz Loops and ice bath

Approximating the thermal conductivity of Copper as a constant $\kappa = 401\,\mathrm{W\,m^{-1}\,K^{-1}}$ (which shows actual variation of a negligable 3% across temperatures $0\,^\circ\mathrm{C}$ to $100\,^\circ\mathrm{C}$) [**water˙density**],

- researched properties of silicon carbide for use as ceramic
- Will estimate maximum allowing temperature of copper (for now assumed 100 °C) by calculating temp at which copper expansion becomes unacceptable: http://www.engineeringtoolbox.com/linear-expansion-coefficients-d_95.html

Analysing the Coil-Conductor interface and Silicon Carbide thermal properties

The rate of heat through we can achieve through the iterface is given by Fourier's law as

$$P = -kA\frac{\Delta T}{\delta x}$$

snag: the contact area of the coil-thermal conductor interface is small, where as the cross-sectional area (that orthogonal to the temperature gradient) of the thermal conductor can be large by our modulation. How can we apply Fourier's law to an abstraction of a single thermal conduction stage?

Email:
I've snagged myself on applying Fourier's Law on the wire - enclosing interface and the enclosing to the bath body (simultaneously, which is probably the issue). P = - k A dT/dx.

That interface contact area is quite small and critical, so I'd definitely want to incorporate it in my calculation of the maximum rate of heat flow across the interface. However, once heat has crossed the interface, it (can) experiences a larger cross-sectional area (surface normal to the temperature gradient) along the temperature gradient (and I, if possible, would want to only calculate a single gradient, rather than equalizing two; one for the body and one for the interface).

I.e. imagine we have a floating sphere from which we want to conduct heat. We can enclose it in a larger spherical shell which has a temperature gradient across its radius. The small sphere to big shell contact area is fixed and all heat flow must go through that area. But the 'cross-sectional area' through which heat can flow to get to the surface of the larger sphere increases over the radius.

Using A = inner sphere surface area would thus (I expect? maybe I'm wrong) present an underestimate of the achieve rate of conduction. It'd be equivalent to having lots of cylinders sticking out of the inner sphere covering its entire surface (of course these cylinders have curved, a-circular faces to have contact the entire sphere surface. In fact I don't know what these weird shapes are. It's probably not even possible by some goofy theorem), where the cylinders have the same temperature gradient across their length as that across the shell's radius.

In the cylinder set up, there's much less volume of cylinder than there is of shell; this probably means the rate of heat transfer it can accommodate is less, right? (although now I'm started to wonder, "why is hot slow?").

so the TL;DR: I'm trying to abstract that there are two stages of thermal conduction(across the interface and through the enclosing body) into a single stage. This is proving tricky with Fourier's Law. What should I do?

for now, I will consider the cross-sectional area of the thermal body to be equal to that of the interface area. Perhaps this will assist fight the overestimation due to the assumption that wires can

PUT ALL THAT ON HOLD

Thermal properties of 6H Silicon Carbide

Reading http://scitation.aip.org/content/aip/journal/jap/50/9/10.1063/1.326720 and http://www.ioffe.ru/SVA/NSM/Semicond/SiC/thermal.html

Questions:

- Does the electrical conductivity become at all significant for temperatures near 100 °C?
- Establish a relationship (or its functional form) between temperature in 293 K to 373 K to thermal conductivity? Na

Lazy screenshot:

Heat Evacuation by Conduction

Over the permitted / expected temperature region of the wires (room temp 20deg to 100deg), the thermal conductor 6H SiC suffers a significant drop of thermal conductivity from 3.9 $\frac{W}{cm\,K}$ to 2.7 $\frac{W}{cm\,K}$. We'll naively apply a linear interpolation. In $\frac{W}{m\,K}$:

In[180]:= $k[T\_] = 10^2 \left(\frac{(2.70 - 3.87)}{100 - 20} T + 3.87 - \frac{(2.70 - 3.87)}{100 - 20} 293\right);$

The outer surface area of the coils around the loops when in a square packing as previously calculated, is

```
In[163]:= A = 0.07539822368615504;
```

and the rate of heat evacuation we must achieve (the heat dissipated into the coils) is

```
In[170]:= P = 425;
```

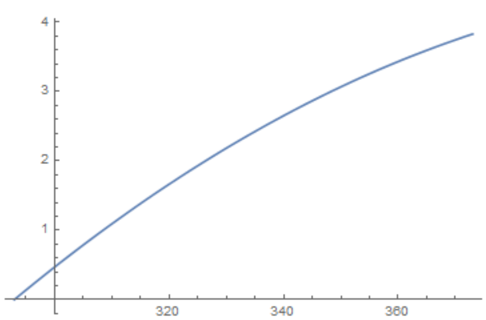
The rate of heat we can evacuate through our interface of area A, down our thermal conductor of length L and temperature dependent thermal conductance k[T] when the wires have temperature Twires is given by Fourier's law as

In[181]:= $equ = P == k[Twires]\, A \frac{Twires - 293}{L};$

We can observe the relationship between the distance we must arrange between the coil-enclosing interface and the ice bath and the maximum temperature we'll permit the wires to reach Twires, by plotting:

```
In[182]:= Plot[ L /. Solve[equ, L], {Twires, 293, 373}]
```

Out[182]=



It turns out the slump in thermal conductivity of 6H SiC is insignificant compared to the possible power output gained from a larger temperature across the gradient, so the maximum distance we can place the ice bath occurs at the max permisseable wire temperature of 100 deg C

This suggests we can, at comfortable wire temperatures (e.g. around 60 degrees) evacuate all heat by conduction when our ice bath is comfy distances from the coils (e.g. around 2 meters).

18/9/15

- added linear pendulum transfer function of g-coupling by force resolutoin to Report Calculations notebook
- explored magnetic field of spherical permanent magnet using mathematica vector plots

Meeting 18/9/15

- Clean up MMA notebook for magnetic field calculations ☑
- Start looking at suspension wire material / thickness and resulting noise profile ☐

18/9/15...

Eep! Email:

RESOURCES FOR DERIVING MAGNETIC FIELD OF CYLINDRICAL MAGNET:

http://rmf.smf.mx/pdf/rmf-e/59/1/59_1_8.pdf
http://www.femm.info/Archives/misc/BarMagnet.pdf
http://web.mit.edu/6.013_book/www/chapter9/9.3.html
https://hal.archives-ouvertes.fr/hal-00354586/document

- formalized spherical magnet field calculation in Report
- Began derivation of cylindrical magnet field

22/9/15

CUT FROM REPORT: attempting to derive cylindrical magnet's field from integration of infiniestimal volume dipoles

The magnetic field around a single, infinitesimal dipole of moment $\vec{\mu}$ located at the origin is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(-\frac{\vec{\mu}}{|\vec{r}|^3} + 3\frac{(\vec{\mu}\cdot\vec{r})}{|\vec{r}|^5}\vec{r}\right),$$

and thus the field due to a single dipole located at $\vec{r}'$ is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(-\frac{\vec{\mu}}{|\vec{r}-\vec{r}'|^3} + 3\frac{\vec{\mu}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^5}(\vec{r}-\vec{r}')\right).$$

To calculate the magnetic field at a point outside of a cylindrical magnet of length L and radius R with uniform magnetization $\vec{M}$, we must sum the contributions of each infinitesimal magnetic dipole moment in the volume of the magnet $\mathcal{V}$ to the field at that point.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\iiint\limits_{\mathcal{V}} -\frac{\vec{\mu}}{|\vec{r}-\vec{r}'|^3} + 3\frac{\vec{\mu}\cdot(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^5}(\vec{r}-\vec{r}')\,\mathrm{d}^3r'.$$

Note that the microscopic dipoles in an actual ferromagnetic are not uniformly directed but have random arrangement and combine to form a spontaneous total magnetization. In our analysis, we treat the total magnetization as being constituted by uniform infinitesimal dipoles. Articulate why this is safe.

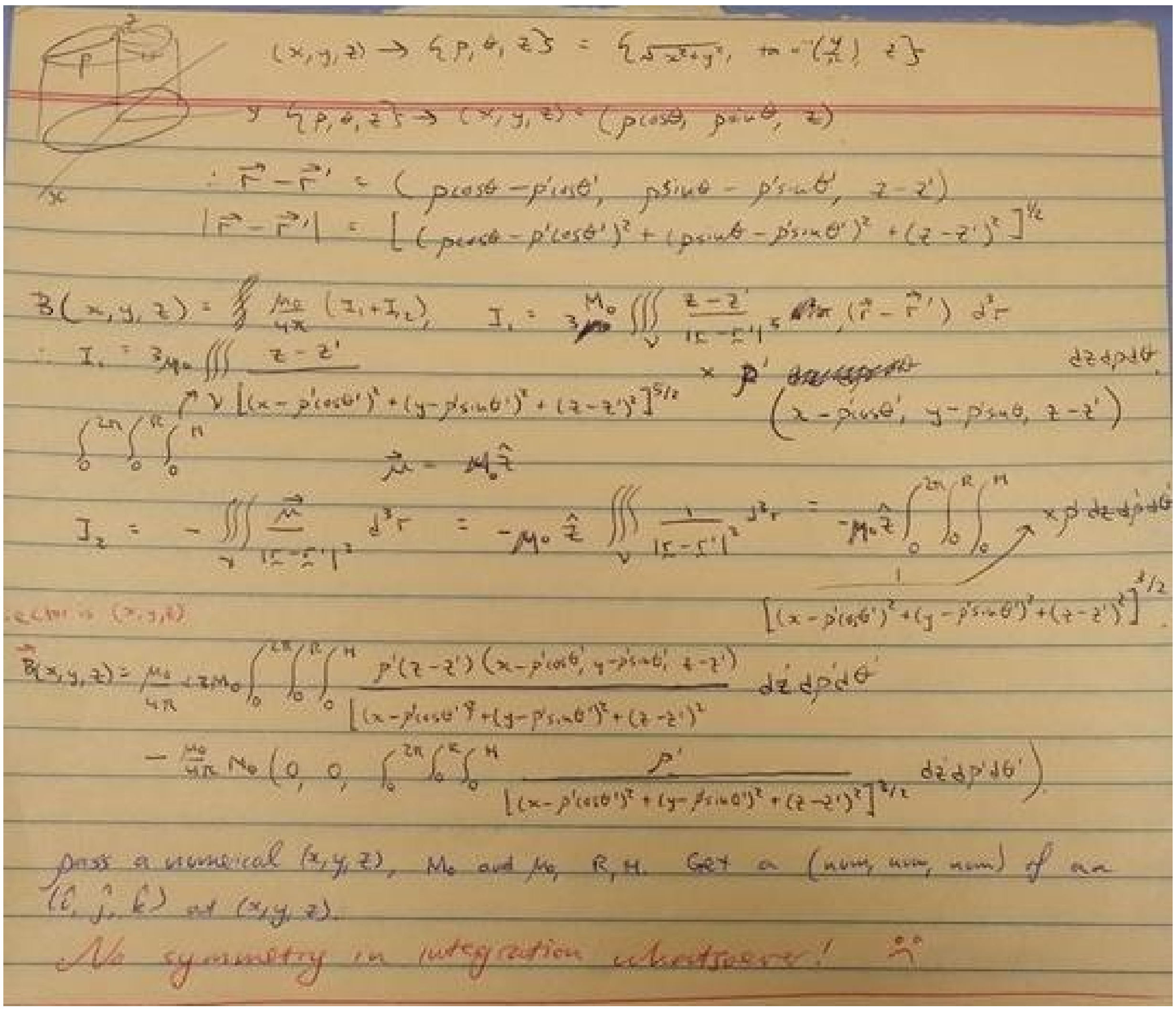
Our cylindrical geometry is best handled with cylindrical coordinates

$$\vec{r'} = (\rho', \theta', z') = \rho'\vec{\rho} + \theta'\vec{\theta} + z'\vec{z},$$

where $\vec{z}$ is parallel to the cylindrical axis, θ is the polar angle and ρ is the normal distance to the axis. In this coordinate system, the magnetic moments are parallel to the z-axis ($\vec{\mu} = \mu\hat{z}$), and therefore

$$\vec{\mu} \cdot (\vec{r} - \vec{r'}) = additionisannoying$$

Simplified in cylindrical coords:



$(x, y, z) \to \{\rho, \theta, z\} = \{\sqrt{x^2+y^2}, \tan^{-1}(\frac{y}{x}), z\}$

$\{\rho, \theta, z\} \to (x, y, z) = (\rho\cos\theta, \rho\sin\theta, z)$

$\therefore \vec{r} - \vec{r}' = (\rho\cos\theta - \rho'\cos\theta', \rho\sin\theta - \rho'\sin\theta', z - z')$

$|\vec{r} - \vec{r}'| = [(\rho\cos\theta - \rho'\cos\theta')^2 + (\rho\sin\theta - \rho'\sin\theta')^2 + (z-z')^2]^{1/2}$

$B(x, y, z) = \frac{\mu_0}{4\pi}(I_1 + I_2)$, $I_1 = 3M_0 \iiint_V \frac{z - z'}{|\vec{r} - \vec{r}'|^5}(\vec{r} - \vec{r}')\, d^3r$

$\therefore I_1 = 3M_0 \int_0^{2\pi}\int_0^R\int_0^H \frac{z - z'}{[(x-\rho'\cos\theta')^2 + (y - \rho'\sin\theta')^2 + (z-z')^2]^{5/2}} \times \rho' (x - \rho'\cos\theta', y - \rho'\sin\theta, z - z')\, dz\, d\rho\, d\theta$

$\vec{\mu} = M_0\hat{z}$

$I_2 = -\iiint_V \frac{\vec{\mu}}{|\vec{r} - \vec{r}'|^3} d^3r = -M_0\hat{z}\iiint_V \frac{1}{|\vec{r} - \vec{r}'|^3} d^3r = -M_0\hat{z}\int_0^{2\pi}\int_0^R\int_0^H \frac{1}{[(x - \rho'\cos\theta')^2 + (y - \rho'\sin\theta')^2 + (z-z')^2]^{3/2}} \times \rho'\, dz'\, d\rho'\, d\theta'$

where $\vec{r}$ is (x, y, z)

$\vec{B}(x, y, z) = \frac{\mu_0}{4\pi} 3M_0 \int_0^{2\pi}\int_0^R\int_0^H \frac{\rho'(z - z')(x - \rho'\cos\theta', y - \rho'\sin\theta', z - z')}{[(x - \rho'\cos\theta')^2 + (y - \rho'\sin\theta')^2 + (z - z')^2]^{5/2}} dz'\, d\rho'\, d\theta'$

$- \frac{\mu_0}{4\pi} M_0 \left(0, 0, \int_0^{2\pi}\int_0^R\int_0^H \frac{\rho'}{[(x - \rho'\cos\theta')^2 + (y - \rho'\sin\theta')^2 + (z-z')^2]^{3/2}} dz'\, d\rho'\, d\theta'\right)$

pass a numerical (x, y, z), M_0 and μ_0, R, H. Get a (num, num, num) of an $(\hat{i}, \hat{j}, \hat{k})$ at (x, y, z).

No symmetry in integration whatsoever!

THIS WAS ABORTED IN LIEU OF USING AN EXISTING SIMPLIFIED EXPRESSION TO BE INTEGRATED NUMERICALLY, from

23/9/15

- Altered small-scale MAGPI params notebook to calculate the mass of the suspended magnet from its spherical radius and the mass-volume density of YIG.

  This resulted in a lighter magnet than expected, pushing the center of mass further down the rod and the sweet spot off the rod; recalculation of other MAGPI parameters were performed.

- Spent the entire day attempting to calculate the magnetic field around a uniformly magnetized cylinder by numerical volume integration

24/9/15

- Integrated calculations of magnet parameters for suspension of the magpi only by external permanent magnets into the PARAMETERS NOTEBOOK (some renaming)
- Compared spherical to cylindrical magnets for external field generations by the volumes required to achieve reasonable magnet fields. Spheres win!
- Analysed the homogeneity of the external magnetic field as generated by a spherical permanent magnet along the dipole axis of the suspended magnet.

  It was thus concluded that the 20cm scale MAGPI CAN be suspended with reasonably sized magnets (spherical magnets of radius 2cm

Meeting 24/9/15

– Calculate the suspending wire thickness to achieve a resonant pendulum frequency of ≈ 10 mHz. (*Hint, explore the use of Tungsten and look up some relevant US Washington Uni experiments* ☑

– Calculate the sizes and geometries of the magnets in the latest MAGPI redesign such that 'flip stability' is achieved. ☐

– Explore the noise budget of the latest MAGPI redesign (*think Fermi questions*) ☐

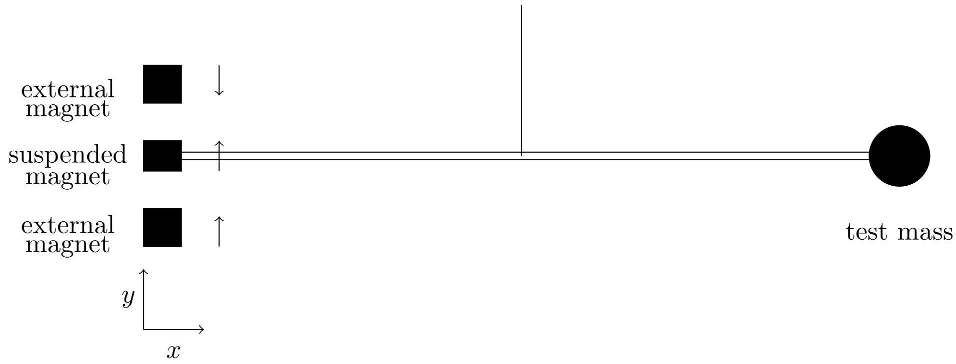
28/9/15

Refer to new MAGPI design sketch (below somewhere).

Considering spherical magnets

Let the remanent magnetic fields of the bottom, middle and top magnets respectively be $(0, B_r, 0)$, $(0, B_r, 0)$ and $(0, -B_r, 0)$ where $B_r > 0$, and let their radii be a_b, a_s, a_t. Their magnetic dipole moments are then $\mu_b = (0, \frac{2\pi {a_b}^3}{\mu_0}, 0)$, $\mu_s = (0, \frac{2\pi {a_s}^3}{\mu_0}, 0)$ and $\mu_t = (0, \frac{2\pi {a_t}^3}{\mu_0}, 0)$.

I want to find the torque on an infinitesimally misaligned-from-antiparallel dipole due to another. I observe 3D rotation symmetry about the dipole, I can plot torque over angle.



CREATED NEW MATHEMATICA DOCUMENT, problem encountered with force calculation.

Note: new magnet location, ignoring negligible rod mass, will double the center of mass' distance from the right end of the bar. This is insignificant.

29/9/15

Too lazy but it's important. Here!

Interaction between suspended and top

The fixed, top magnetic dipole (spherical magnet of radius at and remanent magnetic field Br downward) provides external magetic field

```
Bt[r_, Bin_] = at^3/2 (-1/Norm[r]^3 Bin + 3 (Bin . r)/Norm[r]^5 r)
```

$$\frac{1}{2}\,\text{at}^3\left(\frac{3\,\text{r Bin.r}}{\text{Norm}[\text{r}]^5}-\frac{\text{Bin}}{\text{Norm}[\text{r}]^3}\right)$$

Letting the suspended dipole be distance dt from the top dipole's center, it experiences external field

```
Bext = Bt[{0, -dt, 0}, {0, -Br, 0}] /. Abs → Identity
```

$$\left\{0,\ -\frac{\text{at}^3\,\text{Br}}{\text{dt}^3},\ 0\right\}$$

Let the suspended magnet have radius as and remenant magnetic field (equal in magnitude to the top's) pointing at some angle theta from the horizontal, along the x-y plane

```
μs[θ_] = 2 π as^3/μ0 {Br Cos[θ], Br Sin[θ], 0}
```

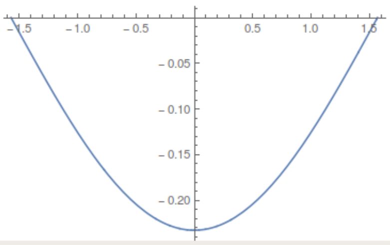
$$\left\{\frac{2\,\text{as}^3\,\text{Br}\,\pi\,\text{Cos}[\theta]}{\mu 0},\ \frac{2\,\text{as}^3\,\text{Br}\,\pi\,\text{Sin}[\theta]}{\mu 0},\ 0\right\}$$

The torque on the suspended magnet is then

```
τt[θ_] = μs[θ] × Bext
```

$$\left\{0,\ 0,\ -\frac{2\,\text{as}^3\,\text{at}^3\,\text{Br}^2\,\pi\,\text{Cos}[\theta]}{\text{dt}^3\,\mu 0}\right\}$$

```
Plot[τt[θ][[3]] /. {as → 1, at → 1, Br → 1, dt → 3, μ0 → 1}, {θ, -π/2, π/2}]
```



and it also experiences force

```
Ft[θ] =
 FullSimplify[Grad[μs[θ] . Bt[{x, y, z}, {0, -Br, 0}] /. Abs → Identity, {x, y, z}] /.
   {x → 0, y → -dt, z → 0}, dt > 0]
```

$$\left\{\frac{3\,as^3\,at^3\,Br^2\,\pi\,\text{Cos}[\theta]}{dt^4\,\mu0},\,-\frac{6\,as^3\,at^3\,Br^2\,\pi\,\text{Sin}[\theta]}{dt^4\,\mu0},\,0\right\}$$

Interaction between suspended and bottom

Note that the origin of the magnetic field is insignificant, since inconsistant between fields. Only consider positions relative to sources, never absolute positions

$$\text{Bb}[r\_,\,Bin\_] = \frac{ab^3}{2}\left(-\frac{1}{\text{Norm}[r]^3}\,Bin + \frac{3\,(Bin\,.\,r)}{\text{Norm}[r]^5}\,r\right);$$

```
τb[θ_] = FullSimplify[ μs[θ] × Bb[{0, db, 0}, {0, Br, 0}], db > 0]
```

$$\left\{0,\,0,\,\frac{2\,ab^3\,as^3\,Br^2\,\pi\,\text{Cos}[\theta]}{db^3\,\mu0}\right\}$$

```
Fb[θ_] =
 FullSimplify[Grad[ μs[θ] . Bb[ {x, y, z}, {0, Br, 0}] , {x, y, z}] /.
   {x → 0, y → db, z → 0}, db > 0]
```

$$\left\{\frac{3\,ab^3\,as^3\,Br^2\,\pi\,\text{Cos}[\theta]}{db^4\,\mu0},\,-\frac{6\,ab^3\,as^3\,Br^2\,\pi\,\text{Sin}[\theta]}{db^4\,\mu0},\,0\right\}$$

Balancing the interactions

Torques

For 'flip' stability we require that within acceptable angle region (i.e. that which be induced by noise: quite small), the torque due to the bottom magnet is larger than or equal to the torque of the top magnet.

```
torquet =
 Norm[τt[θ]] /.
  Abs → Identity (* Cos positive for theta in [-pi/2, pi/2] which is max region*)
```

$$\frac{2\,as^3\,at^3\,Br^2\,\pi\,\text{Cos}[\theta]}{dt^3\,\mu0}$$

```
torqueb = Norm[τb[θ]] /. Abs → Identity
```

$$\frac{2\,ab^3\,as^3\,Br^2\,\pi\,\text{Cos}[\theta]}{db^3\,\mu0}$$

We require (noticing that if satisfied for some angle, it satisfied for all angles)

$$\frac{\text{torqueb}}{\text{torquet}} \geq 1$$

$$\frac{ab^3\,dt^3}{at^3\,db^3} \geq 1$$

Forces

Concern: both bottom and top magnets push the suspended magnet in the direction of its tilt

```
Fb[θ]
```

$$\left\{\frac{3\,ab^3\,as^3\,Br^2\,\pi\,\text{Cos}[\theta]}{db^4\,\mu0},\,-\frac{6\,ab^3\,as^3\,Br^2\,\pi\,\text{Sin}[\theta]}{db^4\,\mu0},\,0\right\}$$

```
Ft[θ]
```

$$\left\{\frac{3\,as^3\,at^3\,Br^2\,\pi\,\text{Cos}[\theta]}{dt^4\,\mu0},\,-\frac{6\,as^3\,at^3\,Br^2\,\pi\,\text{Sin}[\theta]}{dt^4\,\mu0},\,0\right\}$$

7/10/15

Choosing the Suspending Wire

Taking Dr Thrane's recommendation of a Tungsten wire, we seek to determine a suitable radius and length of the suspending wire to achieve a resonant frequency of the system of $f \approx 10\,\text{mHz}$.

The resonant frequency of a torsion pendulum is

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{I}}$$

(stated:http://vlab.amrita.edu/?sub=1&brch=280&sim=1518&cnt=1, derived: http://nano-optics.colorado.edu/fileadmin/Teaching/phys1140/lab_manuals/LabManualM4.pdf) Does this apply when the suspension point ISN'T the center of mass?!?! Consulting the derivation, I is kept general. It's safe to use
where k is the 'rotational stiffness' or more precisely the 'couple per unit twist' (the 'torque causing longitudinal rotation' induced by a twist unit) of the wire.

Tungsten is a linear elastic material (https://en.wikipedia.org/wiki/Young%27s_modulus) and so has a 'shear modulus of elasticity' or 'modulus of rigidity' constant G which relates the shear stress to the shear strain inside the material:

$$\tau = G\gamma$$

(http://ocw.nthu.edu.tw/ocw/upload/8/252/Chapter_3-98.pdf).

For a cylinder of radius r being subjected to an angle of longitudinal twist per unit length θ, the shear strain at a point of radius $\rho < r$ is

$$\gamma = \rho\theta,$$

and so the shear stress at that point is

$$\tau = G\rho\theta.$$

(http://ocw.nthu.edu.tw/ocw/upload/8/252/Chapter_3-98.pdf)

The shear force acting on an infinitesimal area $\mathrm{d}A$ is

$$F = \tau \mathrm{d}A,$$

producing an infinitesimal torque of

$$\begin{aligned}\mathrm{d}T &= F\rho \\ &= \tau\rho\mathrm{d}A\end{aligned}$$

(http://ocw.nthu.edu.tw/ocw/upload/8/252/Chapter_3-98.pdf)

The total torque is thusly

$$\begin{aligned}T &= \int_A \mathrm{d}T \\ &= \int_A \tau\rho\mathrm{d}A \\ &= G\theta\int_A \rho^2\mathrm{d}A.\end{aligned}$$

We can recognise the moment of inertia of a wire (a cylinder):

$$
\begin{aligned}
I_\rho &= \int_A \rho^2 \mathrm{d}A \\
&= \frac{\pi r^4}{2}
\end{aligned}
$$

(http://ocw.nthu.edu.tw/ocw/upload/8/252/Chapter_3-98.pdf)
so our total torque when the wire is subject to an angle of longitudinal twist per unit length θ is

$$T(\theta) = G\theta \frac{\pi r^4}{2}.$$

Assuming the rate of twist along the wire is uniform, then a total twist angle of $\phi = L\theta$ (where L is the length of the wire) produces a total torque of

$$T(\phi) = G\frac{\phi}{L}\frac{\pi r^4}{2}.$$

The 'rotational stiffness' of the wire is then

$$
\begin{aligned}
k &= \frac{T(\phi)}{\phi} \\
&= \frac{G\pi r^4}{2L}
\end{aligned}
$$

(stiffness defined: https://en.wikipedia.org/wiki/Stiffness, derived formula confirmed ($n = G$): http://vlab.amrita.edu/?sub=1&brch=280&sim=1518&cnt=1. The equality of n (modulus of rigidity) and G (shear modulus of elasticity) confirmed: http://www.engineeringtoolbox.com/modulus-rigidity-d_946.html)

We can now sub this into our expression for the resonant frequency:

$$
\begin{aligned}
f &= \frac{1}{2\pi}\sqrt{\frac{k}{I}} \\
&= \frac{1}{2\pi}\sqrt{\frac{G\pi r^4}{2LI}}
\end{aligned}
$$

Taking the modulus of rigidity to of Tungsten to be $G \approx 161\,\mathrm{GPa}$ (agreed by: http://www.engineeringtoolbox.com/modulus-rigidity-d_946.html and https://en.wikipedia.org/wiki/Tungsten) and determining the moment of inertia of our suspended system by the rod parameters (fixed, or rather to be balanced with more difficult system factors, like the external magnetic fields), we can find a relationship between the required thickness and length of the tungsten rope to achieve the required resonant frequency of $f \approx 10\,\mathrm{mHz}$. This relationship can be used to find a reasonable satisfactory length and radius of the suspending wire.

Rearranging,

$$L = \frac{Gr^4}{8f^2 I\pi}.$$

For $G = 161\mathrm{GPa}$, $f = 10 \times 10^{-3}$, $I \approx 0.027\,\mathrm{kg/m^2}$ (found from original MAGPI setup notebook calcs for the reasonable 20cm long MAGPI setup), this turns to

$$L \approx 2.37259 \times 10^6 r^4$$

This calculation excludes the longitudinal moment of inertia of the wire itself, which is negligible compared to that of the rod system.

I'm personally surprised by the thickness of Tungsten wire required when convenient wire lengths are chosen, like a 2 cm radius for 50 cm length. However, volume calculations of the required wire compared with Tungsten price suggested cheap amounts (¡ $30).

Expression of paranoia: Surely these magpi designs have their lateral rotation perturbed by the magnetic fields, which could constitute a perturbed moment of inertia of the rod. I'm merely praying this is insignificant.

8/10/15

Choosing the Suspending Wire...

I had good reason to be suspicious of the thickness of wire required; G was in GigaPascals, and wasn't SI. Letting $G = 161 \times 10^6$ Pa, we calculate

$$L \approx 2.4 \times 10^{12} r^4.$$

See now corrected Figure 3, which suggests a 50 cm length cable would require a radius of 0.6775 mm.

This is definitely achievable (see here: `http://www.goodfellow.com/E/Tungsten-Wire.html`), and it looks like good precision radius is easy to manufacture, though we express slightly concern anyway based on the extraordinarily large growth of Tungsten length required for small growth in radius size.

This is of course the balance for the OLD Magpi design. The difference in motion of inertia between the old and new MAGPI design is worth considering, though should be slight if our test mass is a lot heavier than the rod's mass. Our requirement is that

$$L = \frac{Gr^4}{8f^2 I\pi}.$$

Even drastically doubling our moment of inertia would halve the length for a given radius. To restore the previous length, this would only require a $2^{\frac{1}{4}}$ increase in radius; balancing is therefore easy.

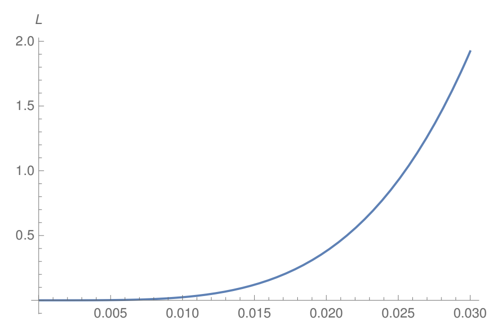


Figure 2: Length vs radius of suspending Tungsten wire to achieve a 10 mHz resonant frequency in the original MAGPI design (moment of inertia $I \approx 0.027\,\text{kgm}^2$)

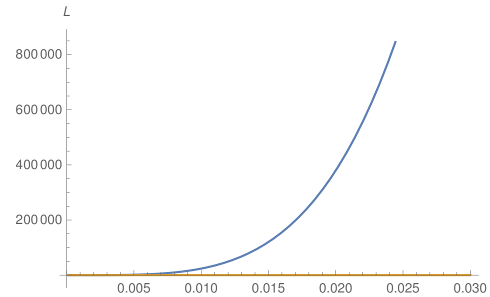


Figure 3: corrected* Length vs radius of suspending Tungsten wire to achieve a 10 mHz resonant frequency in the original MAGPI design (moment of inertia $I \approx 0.027\,\text{kgm}^2$)

I note also that despite the L vs r criticality, actual setup is trivially easy. Once a wire of appropriate radius and length is obtained, the physical setup of the suspension should be such that the length is $l \; r^4$. Although this suggests an incredibly precise l to perfectly balance the radius, the actual sensitivity of the resonant frequency goes like $f \; \frac{1}{\sqrt{L}}$, so greater imprecision in the suspension length is allowed which doesn't strongly affect perturb the resonant frequency. Verification of the frequency requires observation of the torsion pendulum swinging for 100 s, which is ok.

Finally, should we wish to decrease the resonant frequency to 1 mHz (a factor of 10), no adjustment of wire radius would result in a suspension length change of a factor of 100; our 50 cm length cable becomes 5 cm. Instead, adjustment of the radius to which the resonant frequency goes like r^2 simultaneously with the length makes this new frequency easily obtainable. (For example, we can merely shrink our radius by a third - which is achievable, based on this (http://www.goodfellow.com/E/Tungsten-Wire.html), which would yield a resonant frequency one eighty-ninth its original).

New MAGPI Configuration Parameters

CONCERN: does the paper's expression for the sweet-spot apply to this new system?

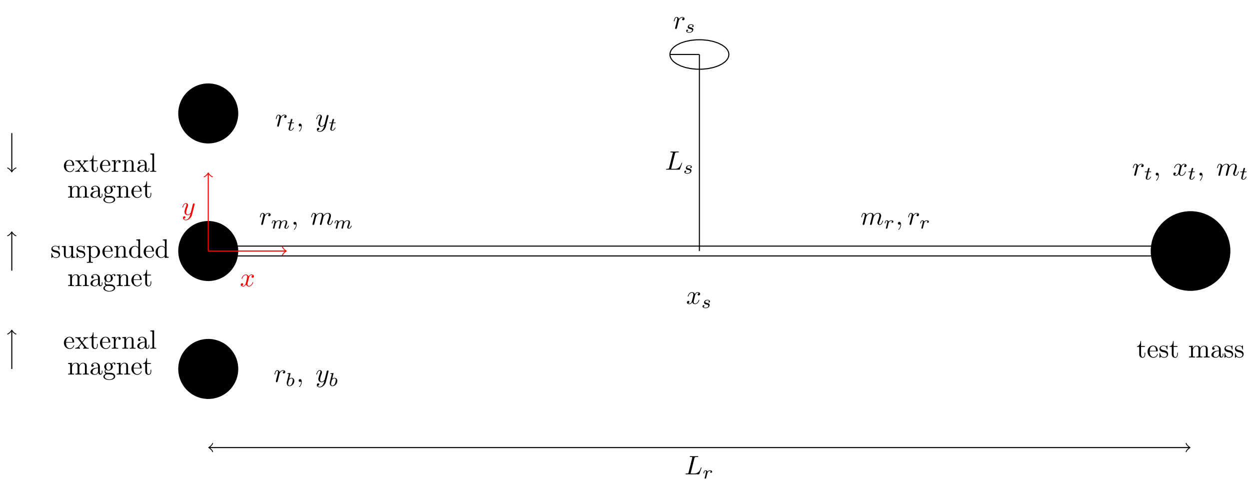


Figure 4: New MAGPI design and parameters

ALERT: a mistake in the original MAGPI's calculation of the sweet-spot was discovered. Achievement

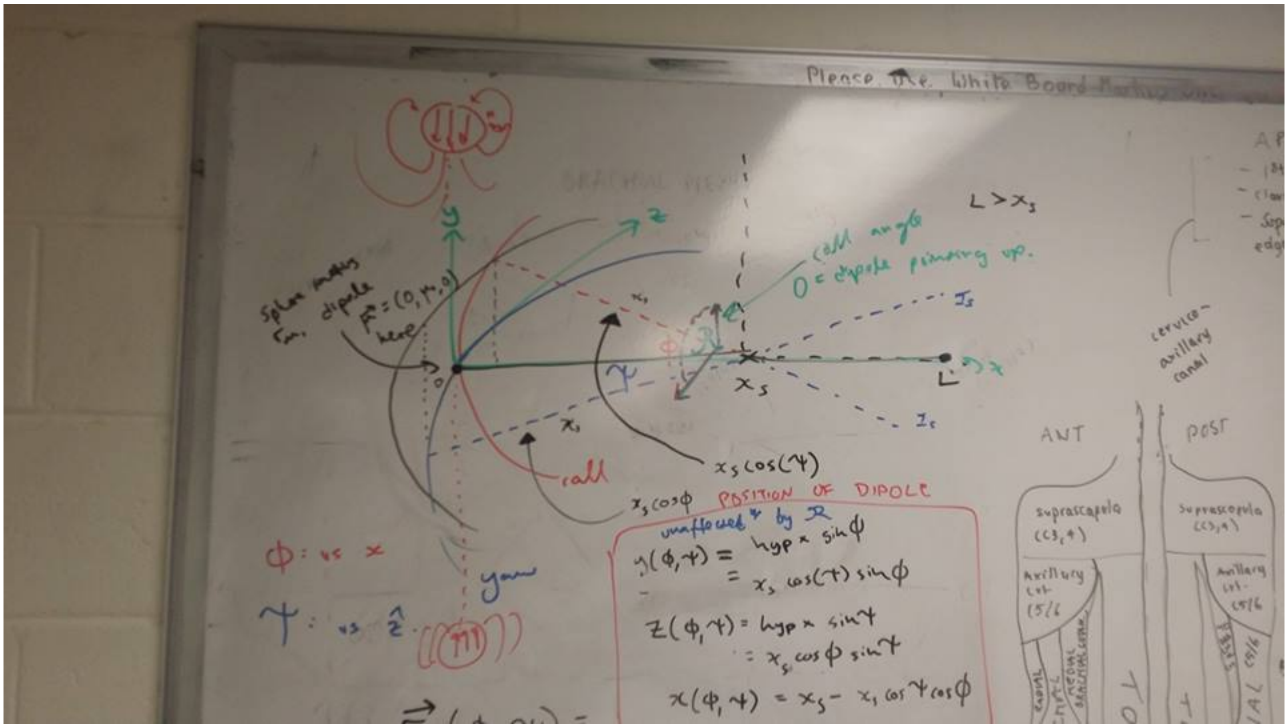


Figure 5: Visualisation of rotation modes caused by noise. Notice that the unlikely roll noise is demonstrated for completeness.

of a convenient sweet-spot for small-scale parameters is now less assured.

9/10/15

- Created a new MMA notebook '8-10-15 new MAGPI parameters.nb' (note hour change since creation date c:) which compiles the adjustable parameters to be balanced (derivations of invoked formula are in separate documents).
- Created a new MMA notebook '8-10-15 precise stability analysis'.

Precise Stability Analysis

Consider Figure 4. Imagine the rod can now pitch, yaw and roll due to noise, forming angles ϕ, ψ and $\mathcal{R}$ respectively (see Figure 5).

When the rod is perfectly parallel to the ground and between the external magnets (though perhaps not at their vertical centre), we say $\phi = \psi = \mathcal{R} = 0$. Here, the fixed magnet is at $\vec{r} = (0, 0, 0)$ and has an upward pointing dipole $\mu\,\hat{y}$.

As geometrically intuited, roll has no affect on the position of the magnet, but pitch and yaw bring the magnet's center to position

$$\vec{r}(\phi, \psi, \mathcal{R}) = \Big(x_s - x_s \cos\phi \cos\psi, \;\; x_s \cos\psi \sin\phi, \;\; x_s \cos\phi \sin\psi\Big).$$

When considering the affect of these angles on the direction of the dipole, we can ignore the fact that rotations are not at the origin; the same direction change is experienced regardless of about which points the rotations are. So, the dipole points along

$$\vec{\mu}(\phi, \psi, \mathcal{R}) = R_x(\mathcal{R})\, R_y(\psi)\, R_z(\phi) \mu \hat{y}$$

EEP! I JUST REMEMBERED THAT ROTATION ORDER MATTERS! TOO TIRED TO COMPREHEND RESULT!

I think roll must be applied first since it is about the axis of the rod which changes direction under the other transformations. Also notice our angle definitions are un-standard, we should be using standard rotation matrices of negative these angles.

$$\therefore \vec{\mu}(\phi, \psi, \mathcal{R}) = R_y(-\psi)\, R_z(-\phi)\, R_x(\mathcal{R}) \mu \hat{y}$$
$$=$$

OOPS! Rotation of ϕ and ψ have a similar issue! I think ϕ should be performed first, which it has been.

NEW DERIVATION OF DIPOLE DIRECTION

Let's first shift the origin to the suspension point. Under some disturbance, the new position of the magnet (formerly at $(-x_s, 0, 0)$) is

$$\vec{r}^*(\phi, \psi, \mathcal{R}) = \Big(-x_s \cos\phi \cos\psi, \;\; x_s \cos\phi \sin\psi, \;\; x_s \cos\psi \sin\phi \Big).$$

The dipole is orthogonal to this vector and uniquely solved by $\mathcal{R}$.

FUDGE IT: returning to rotating initial vector, IGNORING ROLL (there is no roll noise). Note we must apply ϕ rotations before ψ (both restricted to be small), since ψ rotations don't altar initial dipole vector.

$$\begin{aligned}
\vec{\mu}(\phi, \psi) &= R_y(+\psi)\, R_z(-\phi)\, \mu \hat{y} \\
&= \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \begin{pmatrix} \cos(-\phi) & -\sin(-\phi) & 0 \\ \sin(-\phi) & \cos(-\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\psi \, \sin\phi \\ \cos\phi \\ -\sin\phi \, \sin\psi \end{pmatrix}
\end{aligned}$$

In the '8-10-15 precise stability analysis' notebook, precise expressions of the force and torque experienced by the suspended magnet across pitch and yaw of the rod about the suspension point is calculated.

ISSUE: Attempting to attenuate noise in the yaw direction will prevent any gravitational wave coupling! We do expect noise in that direction by the fluctuation dissipation theorem; when we yaw, the suspended vs external magnets experience change in their interaction. This change can also be experienced by vibration of the external and suspended magnets, which therefore result in yaw noise.

Meeting 9/10/15

– Calculate the sizes and geometries of the magnets in the latest MAGPI such that (first order) stable criticality in both angle* and vertical displacement can be achieved (simultaneously), assuming system is symmetric in yaw (external magnets are concentric rings) ☑

– Introduce tensile strength requirements into suspending cable considerations ☑

– Collate MAGPI designs in a presentation with sketches. Attach notes to the presentations for thorough compilation of calculations for supervisors. ☑

10/10/15

Extended the latest MMA notebook to impose conditions on the external spherical magnet parameters (ignoring yaw displacements, to generalise to ring magnets) to achieve a stable, critical displacement at the MAGPI's designed position.

If all magnets were spherical, we notice the presence of a 3D saddle point in magnetic potential around $z = 0$: even if vertical magnetic forces could be balanced, critical instability exists around the yaw axis.

We can counter this by using yaw-symmetric magnetic fields generated by ring magnets; this allows us to also ignore yaw-noise which becomes less significant by the fluctuation dissipation theorem.

For this analysis however, the magnetic field strength along the vertical z axis at $x = y = 0$ will be assumed that due to two spherical external magnets.

(Problems encountered in evaluating derivatives of modulus in Mathematica. Eep!)

13/10/15

Struggled unsuccessfully to clone and push to a new repo on GitLab. Hhhhhhhhhhhhng!

14/10/15

Tensile strength of Tungsten wire

Can our previously calculated wire sustain the MAGPI?

The previous acceptable Tungsten wire parameters were (for a suspension point in the middle of the 20 cm bar) radius $r = 0.00075\,\text{m} = 0.75\,\text{mm}$ (diameter 1.5 mm) and length 50 cm, which gave a resonant frequency of $\approx 10.8\,\text{mHz}$. This is only an order of magnitude larger than the width of a human hair, so its ability to suspend the MAGPI, the newest design of which is estimated around 1 kg, is brought into question.

The yield strength (the force per area withstandable before non-elastic recovery) of Tungsten (of density $19.25 g/cm^3$) is reported (https://en.wikipedia.org/wiki/Ultimate_tensile_strength) as 941 MPa.

As a function of radius of wire r, we require the Tungsten wires (ignoring their elongation when suspended) to achieve a yield strength of at least

$$Y_{\text{strength}}(r) \geq \frac{M_{\text{MAGPI}}\,g}{\pi r^2} \approx \frac{3.2}{r^2}.$$

Imposing the maximum as 941×10^6 Pa, we're limited to using wire radii larger than

$$\begin{aligned} r &\geq \sqrt{\frac{3.2}{941 \times 10^6}} \\ &= 0.00006\,\text{m}. \end{aligned}$$

This is much smaller than our discussed Tungsten wire radius. At our intended $r = 0.75$ mm radius, we exert a pressure of

$$P = \frac{Mg}{\pi r^2} \approx 7.6 \times 10^6$$

on our Tungsten wire. 7.6 MPa is very shy of the Tungsten's yield strength of 941 MPa, so we rest assured that our Tungsten wire won't snap.

Qualifying for Stability

Vertical displacement stability

We hope to find a small region around the desired 'equilibrium' position of the MAGPI rod (parallel to the ground) through which noise may perturb the rod-fixed magnet without a resulting instability of the rod.

Let's assume some reasonable parameters demonstrated wise for the previous MAGPI design, tweaked to compensate for the new setup. For example, let's double the length of the rod but suspend it at its centre, so the distance from suspension point to centre of mass isn't significantly altered.

In particular, consider an 80 g, 40 cm long, 5 mm radius rod suspended about its mid-point. A 170 g, spherical (with radius 2 cm) YIG magnet is fixed at one end, whilst an 800 g test mass is fixed 2 cm in from the other end. For the most part, we will keep these parameters in generality until derivative magnitudes are required estimation.

We aim to calculate the radius and positions of the external spherical YIG magnets to achieve increasingly more difficult degrees of stability. Note our revision of our axis labels illustrated in Figure 6.

Criticality at $(0, 0, 0)$

- Created a new Mathematica notebook '*14-10-15 first order vertical stability analysis.nb*'.

Preliminaries

In[212]:=
```
(* In general, the external magnetic field at position
 r [vector] due to a spherical radius R [scalar] magnet of uniform
 internal magnetic field Bin [vector] located at rmag [vector] is *)
```

In[213]:=
$$\text{Bgeneral}[\text{r\_}, \text{rmag\_}, \text{R\_}, \text{Bin\_}] = \frac{R^3}{2}\left(-\frac{1}{\text{Norm}[r - \text{rmag}]^3}\text{Bin} + \frac{3\,(\text{Bin}\,.\,(r - \text{rmag}))}{\text{Norm}[r - \text{rmag}]^5}(r - \text{rmag})\right);$$

In[214]:=

In[215]:=
```
(* The magnetic dipole moment of a spherical magnet of radius
 R [scalar] and uniform internal magnetic field Bin [vector] is *)
```

In[216]:=
$$\mu\text{general}[\text{R\_}, \text{Bin\_}] = \frac{2\,\pi\,R^3}{\mu 0}\text{Bin};$$

In[217]:=

In[218]:=
```
(* The force on a dipole μ2 caused by a dipole μ1 when the former
 dipole is a displacement dr [vector] from the latter dipole is *)
```
$$\text{Fgeneral}[\text{dr\_}, \mu\text{1\_}, \mu\text{2\_}] = \frac{3\,\mu 0}{4\,\pi\,\text{Norm}[\text{dr}]^5}\left((\mu 1\,.\,\text{dr})\,\mu 2 + (\mu 2\,.\,\text{dr})\,\mu 1 + (\mu 1\,.\,\mu 2)\,\text{dr} - \frac{5\,(\mu 1.\text{dr})\,(\mu 2.\text{dr})}{\text{Norm}[\text{dr}]^2}\text{dr}\right);$$

In[219]:=

```
(*
physical constants,
μ0 → vacuum permitivity,
g → gravity acceleration near Earth's surface,
dYIG → mass density of YIG,
BYIG → internal (uniform) mag field strength of YIG
*)
```
$$\text{constants} = \{\mu 0 \to 4\,\pi\,10^{-7},\ g \to 9.8,\ \text{dYIG} \to 5172,\ \text{BYIG} \to 1\};$$

In[220]:=

Building the system

MAGNETISM

In[221]:=

In[222]:=
```
(* Consider the 'suspended' magnet (radius rm) to naturally (desirably) reside
 at (0, 0, 0). External magnets sit above and below this (vertically);
a 'top' magnet of radius rt at (0, 0, zt) and a '
 bottom' magnet of radius rb at (0, 0, zb). *)
```

In[223]:=
```
post = {0, 0, zt};
posm = {x, y, z};
posb = {0, 0, zb};
```

In[226]:=

In[227]:=
```
(* The system has dipoles... *)
```

In[228]:=
```
μt = μgeneral[rt, {0, 0, -BYIG}];
μm = μgeneral[rm, {0, 0, +BYIG}];
μb = μgeneral[rb, {0, 0, +BYIG}];
```

In[285]:=

```
(* Physical impositions *)
assumptions = {zb < 0 && zt > 0 && BYIG > 0 && rm > 0 && rb > 0 &&
    rt > 0 && μ0 > 0 && 0 < xs < Lr && Lr > 0 && xt > 0 && mt > 0 && mr > 0};
```

In[232]:=

In[233]:=
```
(* The total force on the middle (suspended)
 dipole located at position (x, y, z) is *)
Ftotal[x_, y_, z_] = Fgeneral[posm - post, μt, μm] + Fgeneral[posm - posb, μb, μm];
```

In[234]:=
```
(* Let's confine ourselves to considering the force when we're perfectly
 aligned with the axis of separation of the magnets (along 0, 0, z). *)
```

In[235]:=
```
F[z_] = Simplify[F[0, 0, z], assumptions && {zb < z < zt}][[3]];
```

In[236]:=

In[237]:=
```
(* The 'magnetic' torque about the suspension
  point (located at xs along the rod length Lr) is then... *)
τm[z_] = F[z] xs;
```

In[238]:=

GRAVITY

In[239]:=
```
(*
suspended magnet's mass,
TOTAL rod's mass (mr is rod mass, mt is test mass),
center of mass along rod (Lr is rod length, xt is test mass position along rod)
*)
```

In[240]:= $mm = \frac{4}{3}\pi\, rm^3\, dYIG;$

In[241]:= `M = mm + mr + mt;`

In[242]:= $xc = \frac{1}{M}\left(mr\frac{Lr}{2} + mt\, xt\right);$

In[243]:=

```
(* meanwhile, gravity applies a (approximated constant over small angles)
  force at the center of mass xc of... *)
τg = (xc - xs) g M;
```

In[244]:=

Imposing Stability

In[245]:=
```
(* For general parameters,
the TOTAL torque about the suspension point of the system as
  a function of the suspended magnet's veritcal position z is... *)
```

In[266]:=
```
τtotal[z_] = τm[z] - τ;    (* don't sub τg yet, for simplicity*)
```

In[247]:=
```
(* We now consider z ONLY between zb and zt *)
```

In[268]:=
```
τtotal[z_] = FullSimplify[τtotal[z], z < zt && z > zb]
```

Out[268]= $-\frac{6\, BYIG^2\, \pi\, rm^3\, xs\left(rt^3\,(z-zb)^4 + rb^3\,(z-zt)^4\right)}{(z-zb)^4\,(z-zt)^4\,\mu 0} - \tau$

In[250]:=
```
(* We first require ZERO TORQUE at the desired equilibrium
 position (z = 0, so the rod is parallel to the ground) *)
```

In[278]:=
```
req1 = FullSimplify[req1 , assumptions]
```

Out[278]= $6\, BYIG^2\, \pi\, rm^3\, xs\left(rt^3\, zb^4 + rb^3\, zt^4\right) + zb^4\, zt^4\, \mu 0\, \tau == 0$

In[253]:=

```
(* We secondly require that this point is STABLE: when z > 0,
the torque is negative and when z < 0,
the torque is positive. This implies our gradient at 0 be negative *)
```

In[275]:=
```
req2 = Simplify[ τtotal'[0]  < 0, assumptions]
```

Out[275]= $rt^3\, zb^5 + rb^3\, zt^5 < 0$

```
(* We also want the negative gradient to apply over some non-zero domain of z,
which will form our noise threshold tolerance. We want to know the size of
 this domain. We want to see for what z's the following is satisfied *)
```

In[299]:=
```
req3 = Simplify[ τtotal'[z] < 0, assumptions && z > zb && z < zt && μ0 > 0 &&
   BYIG > 0 && rm > 0 && xs > 0 && rb > 0 && rt > 0 && zb < 0 && zt > 0]
```

Out[299]= $rt^3\, (z - zb)^5 + rb^3\, (z - zt)^5 > 0$

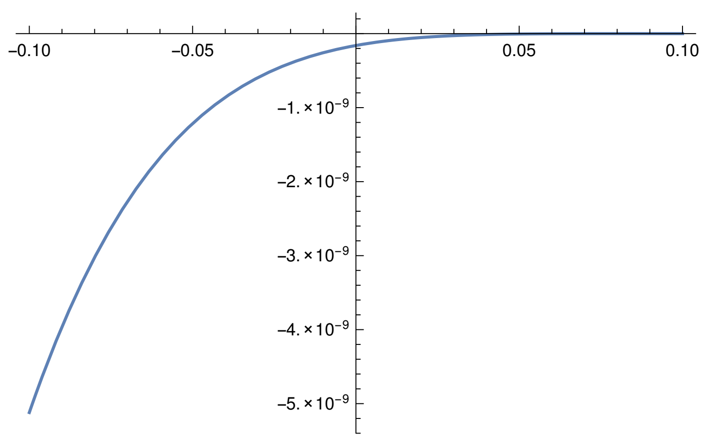
In[303]:=
```
Solve[req3, z]
```

Solve::fdimc : When parameter values satisfy the condition
((rt | zb | zt) ∈ Reals && rb > Root[rt^3 + Slot[≪1≫]^3 &, 1]) || ((rt | zb | zt) ∈ Reals && rb < Root[rt^3 + Slot[≪1≫]^3
&, 1]), the solution set contains a
full-dimensional component; use Reduce for complete solution information. ≫

Out[303]= { }

In[300]:=
```
Plot[ rt^3 (z - zb)^5 + rb^3 (z - zt)^5  /.
  {rt → 0.02, zb → 0.1, rb → 0.02, zt → 0.1}, {z, -0.1, 0.1}]
```

Out[300]=



15/10/15

- Begun presentation of MAGPI designs.

Work flow snag: was creating the designs in TikZ to be used in MathTex Plugin in LibreOffice. The plugin can't handle some geometry libraries used.

I then tried to import the LaTeX generated pdf (trimmed) into LibreOffice Impress or Drawing but it is rendered incorrectly. Woe is me.

Below is a wasted design which can't be integrated into LibreOffice Impress.

- Switched to building designs in Inkscape.
- Created diagrams

16/10/15

Algebraic conditions for stability (ERRONEOUS)

Consider placing the external YIG magnets (of internal magnetisation BYIG) at $(0, 0, z_t)$ and $(0, 0, z_b)$ $(z_t > 0, z_b < 0)$ with radii $r_t > 0$ and $r_b > 0$. The suspension point is located x_s along the axis of the rod. The suspended magnet has radius $r_m > 0$. The torque produced by the external magnetic fields about the suspension point aims to counter the gravitational torque of τ_g.

Our first condition is that at the desired $z = 0$ position of the suspended magnet, the net torque is zero.

$$6\,\mathrm{BYIG}^2\,\pi {r_m}^3 x_s({r_t}^3{z_b}^4 + {r_b}^3{z_t}^4) + {z_b}^4{z_t}^4\mu_0\tau_g = 0.$$

We secondly require that the point is stable; the first derivative of the net torque at $z = 0$ is negative such that a change in z receives a restoring force.

$${r_t}^3{z_b}^5 + {r_b}^3{z_t}^5 < 0.$$

We extend the second requirement over a domain of z, requiring the derivative stay negative (the torque remaining restoring) over a small region of z identified by...

$${r_t}^3(z - z_b)^5 + {r_b}^3(z - z_t)^5 > 0.$$

We seek the domain of z about 0 which satisfies this.

Numerical conditions for stability

Let's assume the rod parameters which have shown success in achieving other previous physical conditions, listed in Table 1.

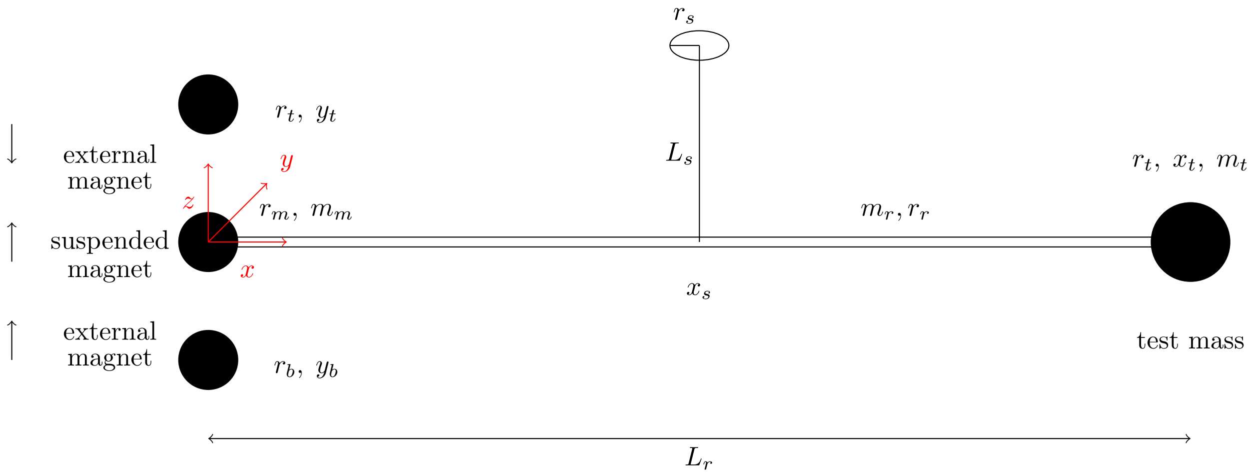


Figure 6: New MAGPI design and parameters (revised axis)

We also substitute constants $\mu_0 = 4\pi 10^{-7}$, $g = 9.8$, $dYIG = 5172$ and $BYIG = 1$ (the mass density and remanent magnetisation of YIG respectively).

This presents a numerical form of our first condition (that the equilibrium position of zero net torque is at $z = 0$):

$$r_t{}^3 z_b{}^4 + (r_b{}^3 + 0.0446459\ z_t{}^4) = 0.$$

With dismay, we notice this has no real solutions for $r_t, r_b > 0$ and $z_t > 0$, $z_b < 0$. Looking at the algebraic requirement, this was evidently *not* due to our use of previous parameters; inspection of the algebraic requirement shows the only negative term z_b is to an even power ($z_b{}^4$) so the equation features two strictly positive terms summing to zero.

By how can a zero net torque be true? For any given gravitational torque, I can surely steadily increase the magnetic torque until it balances! It MUST be balanaceable!!

Suspicion: Some torque term is signed incorrectly such that magnetic and gravitational torques don't oppose.

- Sign of magnetic torque discovered incorrectly negative.

Since the magnetic force on the left side of the rod points downward, its value was negative, but the torque when calculate was not taken to be positive. That is, the displacement from the suspension point $-x_s$ was taken as positive incorrectly.

This was corrected.

Algebraic conditions for stability

Having corrected our magnetic torque sign, the new algebraic conditions are...

Net torque at $z = 0$ is zero:

$$6\ \mathrm{BYIG}^2 \pi r_m{}^3 x_s (r_t{}^3 z_b{}^4 + r_b{}^3 z_t{}^4) = z_b{}^4 z_t{}^4 \mu_0 \tau_g.$$

This happily yields real solutions for z_b for example.

We next require the derivative of net torque at $z = 0$ is negative:

$$ {r_t}^3 {z_b}^5 + {r_b}^3 {z_t}^5 < 0. $$

The threshold of tolerable z values (to which noise may perturb us, assuming we have negligable velocity there) in which stability is retained (between external magnets) is where z satisfies

$$ {r_t}^3 (z_{z_b})^5 + {r_b}^3 (z - z_t)^5 < 0. $$

Numerical conditions for stability

Condition of $z = 0$ having zero net torque:

$$ (-22.40 {r_b}^3 + {z_b}^4) {z_t}^4 = 22.40 {r_t}^3 {z_b}^4. $$

Although we must only consider solutions to this when the subsequent conditions are also satisfied, this is easy to satisfy! For example, letting the external magnets have radius $r_t = r_b = 1\,\text{cm}$ (and the suspended magnet a radius of $2\,\text{cm}$), then magnet placements of $z_b = -0.080\,\text{cm}$ and $z_t = 0.084\,\text{cm}$ will allow the fields to balance.

The condition of local stability (derivative is negative at $z = 0$), when the top and bottom external magnets have the same radius, is

$$ z_t > |z_b|. $$

This was already satisfied by our previous condition parameters, and is easy to establish.

Finally, we seek the z domain across which stability is maintained. Imposing the external magnets to share radii, this becomes

$$ (z - z_b)^5 < (z_t - z)^5. $$

This is satisfied when the distance of the suspended magnet to the top magnet is larger in magnitude than the distance to that and the bottom. I.e. stability is maintained whenever the suspended magnet is closer to the bottom magnet.

THIS CAN'T BE THE FULLL STORY??? WAT

Mathematica produced satisfaction when:

$$ z < \frac{z_b + z_t}{2}. $$

Meeting 16/10/15

- Add introduction slide to presentation (gravity waves & LIGO) ☐
- Remove spherical external magnet forced MAGPI slide ☐
- Add slide with compilation of involved Physics ☐
- Check stability analysis ☐
- Check Cylindrical vs Spherical magnet analysis ☐

Personal complications sunk a lot of time here

25/10/15

- Developed some more diagrams for the oral presentation
- Researched the astrophysical sources of gravitational waves

26/10/15

- Created a new document '*PHS3350 - Design Calculations*' to rigurously present calculations performed on each aspect of each design (really for supervisor's convenience).

27 - 30/10/15

- Formalised new MAGPI stability analysis
- Developed oral presentation
- Created new MMA notebook '*29-10-15 Sweet Spot Calculator*', formalising sweet-spot calculations

Discoveries

- Initial calculations of cable length and thickness were off (by 10^3) since gigapascals were mistaken for megapascals. Correction isn't expected too detrimental since required cable length (which grew by 10^3) can be reduced by radius of sensitivity r^4, and wire pressure (which must be kept below yield strength) goes only like r^2.

Parameter	Symbol	Value	Unit
rod mass	m_r	80	g
rod length	L_r	40	cm
rod radius	r_r	5	mm
suspension point	x_s	20	cm
test mass	m_t	800	g
test position	x_t	38	cm
test radius	R_t	4	cm
suspended magnet radius	r_m	2	cm

(a) Basic rod parameters

Property	Symbol	Value	Unit
suspended magnet mass	m_m	173	g
total system mass	M	1.05	kg
centre of mass	x_c	0.30	cm
moment of inertia (about x_c)	I_c	0.023	kgm^2
moment of inertia (about x_s)	I_s	0.034	kgm^2
gravitational torque	τ_g	1.07	Nm

(b) Resulting physical properties

Table 1: Affects of basic rod parameters